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Production Uncertainty, Enforcement and Smuggling: A Stochastic Model

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PRODUCTION UNCERTAINTY, ENFORCEMENT
AND SMUGGLING:

A STOCHASTIC MODEL

BY

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ABSTRACT

This paper merges the existing smuggling literature with the literature concerning competitive firm behavior under uncertainty. A stochastic, joint product, model of smuggling is developed. The model introduces production uncertainty, generated by enforcement activity, into a Pitt (1981) type of smuggling production function. This modelling technique allows the trade pattern and welfare results which evolve under smuggling with uncertainty to be compared to smuggling in a world of certainty. It is demonstrated that mere presence of uncertainty increases the real resource cost associated with smuggling, reduces legal and illegal trade, and welfare, when compared to smuggling in a world of certainty.

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INTRODUCTION

The literature on illegal transactions in international trade has expanded rapidly since the publication of the seminal article by Bhagwati and Hansen (1973). The smuggling literature can be categorized as presenting either: 1) deterministic models; or 2) stochastic models.

The model presented in this paper is a joint product stochastic model of smuggling. Joint product smuggling was first considered by Pitt (1981). Pitt's model is deterministic and was the first to demonstrate that legal and illegal trade could coexist with the empirically valid phenomena of price disparity. Martin and Panagariya (1984) were the first to formally incorporate uncertainty into a model of joint product smuggling.[1] They introduce uncertainty through the use of a linear probability function, applied externally to the smuggling firm's profit function. Martin and Panagariya were able to reproduce the welfare results of Bhagwati and Hansen, as well as those produced by Pitt, depending on the assumptions imposed. Another important contribution of their paper was the first explicit examination of the effects enforcement has on smuggling and welfare.

This paper extends the literature on smuggling under uncertainty by transforming Pitt's model of smuggling into a stochastic model of smuggling in the tradition of Sandmo (1971), Batra and Ullah (1974), and Ratti and Ullah (1976). Uncertainty is introduced in the smuggling production function via a

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[1] Other papers employing stochastic models include, Schöler (1989), Sheikh (1989), Thursby (1991), Fausti (1992). However, all of these papers fail to provide a link between smuggling under uncertainty with the literature on the economic consequences of the competitive firm operating in a world of uncertainty.

random variable, the smuggling success rate. The smuggling success rate is assumed to be dependent on the level of enforcement effort against smuggling.

The stochastic model presented in this paper allows an analysis of how uncertainty affects the smuggling firm's input demand and output supply decisions. Unlike the earlier papers on smuggling under uncertainty, the firm's input demand and output supply decisions and subsequent welfare consequences can be compared to smuggling in a world of certainty. The stochastic modelling techniques applied in the analysis of smuggling for this paper bridges the gap between the smuggling literature and the literature on "the competitive firm under uncertainty".

II. PITT'S MODEL OF SMUGGLING

Pitt's model lends itself to the introduction of uncertainty through his smuggling production function. Pitt's original trade pattern and welfare results will be compared to the results derived in the uncertainty model, to determine the economic effects of uncertainty in the production of a joint-product export.

Pitt's basic model represents the small country case with fixed terms of trade. The country produces two traded goods, (X) and (M), an exportable and importable, respectively, with primary factors in perfect competition. Production and trade are carried out by identical firms. Legal and illegal trade in exports is carried out by the same firm. All firms smuggle in the Pitt model and the law of one price holds in the domestic economy. It is assumed each firm can trade illegally according to "Pitt's smuggling function",

$$S^* = G(L,S). \tag{1}$$

The term (S^*) is the quantity of good (X) successfully smuggled, (L) is the quantity of good (X) legally traded and (S) is the quantity of good (X) used as an input into smuggling activity. The function (G) is strictly concave and a twice differentiable linear homogenous function. The function (G) is also assumed to have the following properties:

$$G_L \geq 0, \tag{2}$$

$$1 \geq G_S \geq 0, \tag{3}$$

$$S - S^* \geq 0. \tag{4}$$

Assumption (2) states that the marginal product of legal trade used in smuggling is non-negative. Assumption (3) states that a unit increase in the smuggling input (S) results in a positive, but less than or equal to, unit increase in actual ex-post smuggling. Assumption (4) prohibits the cost of smuggling from being negative. The difference between ex-ante smuggling (S) and ex-post smuggling, S^*, is the real resource cost or the confiscation cost of smuggling or both.

Maximization of smuggler's profits is given by equation (5),

$$\pi = P^f \cdot G(L,S) + P^f \cdot (1-t) \cdot L - P^S \cdot (L+S) \tag{5}$$

where [$P^f \cdot G(L,S)$] represents successful smuggling revenues and $P^f \cdot (1-t) \cdot L$ represents revenues for legal trade and $P^S \cdot (L+S)$ represents the domestic production cost of exports. The first order profit maximization conditions, $\partial\pi/\partial L$, $\partial\pi/\partial S$, are respectively,

$$P^f \cdot G_L + P^f \cdot (1-t) = P^S, \tag{6}$$

$$P^f \cdot G_S = P^S. \tag{7}$$

The term (P^f), is the fixed international terms of trade and (t) is the ad valorem tax rate. First order conditions (6) and (7) state that the marginal cost of an additional unit of tradeable will just equal its revenue in trade,

be it legal or illegal trade. An additional unit of legal trade will result in additional legal revenue $P^f \cdot (1-t)$ and additional smuggling revenue $P^f \cdot G_L$. Under the assumption of perfect competition, firms will earn zero economic profits because the revenue from all foreign trade is just equal to the domestic cost of exports. Setting equation (5) equal to zero and solving for P^S yields an expression for the long-run equilibrium domestic price ratio as a weighted average of all export trade,

$$P^S = [P^f \cdot S^* /(L+S)] + [P^f \cdot (1-t) \cdot L /(L+S)]. \quad (8)$$

Equation (8) is equivalent to Pitt's expression for the equilibrium domestic price ratio in his model. Examining eq.8, a price differential is found when comparing P^S to $P^f(1-t)$. Pitt calls this empirically valid price differential, *price disparity*.

Pitt's smuggling production function, eq. 1, embodies a technology that requires the use of both legal and illegal goods as inputs to produce a successfully smuggled good as part of a unit of a joint product tradeable. The smuggling technology requires that a portion of the illegal good (input) will be used up during the transformation process. According to Pitt, during the smuggling process some of the smuggling input is lost to confiscation or some is lost to the real resource cost in excess of legal trade associated with smuggling or both. Therefore, the successfully smuggled good (output) is less than the illegal input. This "using up" of a portion of the illegal input due to an excessive real resource cost is referred to as the Samuelson "melting ice effect", by Bhagwati and Hansen.[2] In the Pitt model, if there is a melting ice effect, then smuggling has an ambiguous welfare effect as

[2] See Samuelson's 1954 paper on trade impediments and transportation costs.

compared to the non-smuggling situation. If the portion of illegal input lost during the smuggling process is the result of confiscation by enforcement activity, then smuggling has a strictly positive welfare effect.

III. SMUGGLING TECHNOLOGY

The purpose of the smuggling technology is to avoid detection so that illegal exports are successfully smuggled out of the country and delivered to the world market as (S^*). As in Pitt's model, it is assumed that the smuggling production function embodies a melting ice effect due to a real resource cost associated with the smuggling technology. The smuggling technology requires smugglers to engage in evasion tactics.[3] In contrast to Pitt's model, the smuggling production function used in this paper is assumed not to embody the confiscation cost associated with smuggling. It is also assumed, that if smugglers do not engage in evasion tactics, then the probability of detection is one.

The smuggling technology embodied in the smuggler's production function, however, does not completely insulate the smuggler from detection and confiscation. Thus, the contribution of the smuggling input (S) to successful smuggling (S^*) during the transformation process via the smuggling technology is reduced by enforcement effort.

Assume the smuggling production function defined in equation (1) now embodies the smuggling technology described above. Define the parameter (u), as the smuggling success rate, having a value between zero and one. Let S^1,

[3] The real resource cost can be due to special packing cost necessary to hide smuggled goods or the excess transportation costs of shipping unreported production out of the country via clandestine ports.

be defined as the amount of smuggling input that avoids confiscation during the smuggling (production) process. Define the relationship between S and S^1 as follows,

$$S^1 = u \cdot S. \tag{9}$$

This assumption allows the replacement of S with S^1 in the smuggling production function,

$$S^* = G(L,S^1). \tag{10}$$

If the smuggling success rate is equivalent in equations 1 and 10, then this modification will not alter Pitt's equilibrium domestic price ratio, eq.8. This implies that separating the real resource cost from the confiscation cost of smuggling has no effect on S^*. The first order conditions (FOC) are derived with the modification,

$$\pi = P^f \cdot G(L,S^1) + P^f \cdot (1-t) \cdot L - P^S \cdot (L+S), \tag{11}$$

$$P^f \cdot G_L + P^f \cdot (1-t) = P^S \tag{12}$$

$$P^f \cdot G_{S1} \cdot dS^1/dS = P^S, \text{ where } dS^1/dS=u \text{ and} \tag{13}$$

$$S > S^1 > S^*. \tag{14}$$

The modified smuggling production function now only generates the real resource effect described in Pitt's paper. The implications of condition 14 are: 1) the confiscation cost associated with smuggling is equal to $S-S^1$ or $S \cdot (1-u)$; 2) the real resource cost associated with smuggling is equal to S^1-S^*; and 3) an increase in u, increases the firms profit, i.e., $\partial\pi/\partial u = S \cdot G_{S1} > 0$.

IV. UNCERTAINTY IN THE SMUGGLING PRODUCTION FUNCTION

In the previous section, the real resource cost component and the confiscation cost component of smuggling were separated by redefining the technology of smuggling and introducing the smuggling success rate (u) into the smuggling production process. Employing a variant of a modeling procedure developed in a paper by Ratti and Ullah (1976), the smuggling success rate will now be defined as a random variable. Define (u) as a strictly positive random variable with the variable's density function defined as f(u),

$$\int_0^1 f(u)\,du = 1. \qquad (15)$$

This modification will make the confiscation cost associated with smuggling a random variable, and introduces uncertainty into the smuggling production function. The implication is that output (S^*), the successfully smuggled good, is now a random variable. The incorporation of uncertainty necessitates redefining the properties of smuggling production function so that it is: 1) a thrice differentiable, concave, non-linear homogenous function; 2) the marginal products are positive and declining; and 3) the cross partial derivatives are positive.[4] The introduction of uncertainty into the smuggling production function leads to the first proposition in the paper:

PROPOSITION I: ***The firm's (expected) output of successful smuggling when uncertainty is introduced, ceteris paribus, is less than the firm's output under conditions of certainty.***

To establish the above proposition, Jensen's inequality and the definition of expected value are applied to the smuggling production function

[4] The assumption of the cross partial being positive implies legal and illegal inputs are complementary.

$G(L,S^1)$. Defining the expected value of (u) as $E(u)=\hat{u}$, where E is the expectations operator; certainty in this situation means to replace (u) with $(\hat{u})$. Then by the Jensen Inequality,

$$E[G(L,uS)] < G(L,\hat{u}S), \tag{16}$$

and proposition I is established.[5]

An economic implication of the introduction of production uncertainty into Pitt's model is that the mere presence of uncertainty, ceteris paribus, increases the real resource cost of smuggling as compared to the case where the firm is operating in a world of certainty.

The introduction of uncertainty into the smuggling production function requires that the smuggling firm's profit function (eq.11) be redefined in terms of utility. It is assumed that the firm's utility function conforms to the characteristics of a von Neumann-Morgenstern utility function and the third derivative of the utility function exists. Under these assumptions, the firm's utility of profit function is,

$$U = U(\pi) = U[P^f \cdot G(L,S^1) + P^f \cdot (1-t) \cdot L - P^S \cdot (L+S)], \tag{17}$$

with $U'(\pi)>0$, and $U'' \gtreqless 0$, depending upon whether the firm is risk preferring, risk neutral, or a risk averter.

It is assumed that the firm's goal is to maximize expected utility from profit. The first order conditions are,

$$\partial E[U(\pi)]/\partial L = E[U'(\pi) \cdot (P^f \cdot G_L + P^f \cdot (1-t) - P^S)] = 0, \tag{18}$$

$$\partial E[U(\pi)]/\partial S = E[U'(\pi) \cdot (P^f \cdot G_{S1} \cdot u - P^S)] = 0. \tag{19}$$

[5] The Jensen Inequality states that if a function is concave the following is true: $E[h(X)] < h(E[X])$. The implication for the smuggling technology is that the output of successful smuggling S^* in an uncertain environment is less than the output of S^* if production had taken place with the expected value of the random variable S^1, i.e., a certain environment. See Rao (1973, p.58) for an explanation of Jensen's inequality.

The second order conditions are,

$$\partial^2 E[U(\pi)]/\partial L^2 = A_1 = E[U''(\pi)\cdot(P^f\cdot G_L + P^f\cdot(1-t)-P^S)^2 + P^f\cdot G_{LL}\cdot U'(\pi)] < 0, \tag{20}$$

$$\partial^2 E[U(\pi)]/\partial S^2 = A_2 = E[U''(\pi)\cdot(P^f\cdot G_{S1}\cdot u - P^S)^2 + P^f\cdot G_{S1S1}\cdot u^2\cdot U'(\pi)] < 0, \tag{21}$$

$$\partial^2 E[U(\pi)]/\partial L\partial S = B_1 = E[U''(\pi)\cdot(P^f\cdot G_L + P^f\cdot(1-t)-P^S)\cdot(P^f\cdot G_{S1}\cdot u - P^S) + U'(\pi)\cdot P^f\cdot u\cdot G_{LS1}], \tag{22}$$

where it is assumed that,

$$A_1\cdot A_2 - B_1{}^2 = DET > 0. \tag{23}$$

V. THE EFFECT OF UNCERTAINTY ON LEGAL AND ILLEGAL TRADE.

In this section it will be demonstrated that the introduction of uncertainty into the Pitt smuggling production function will generate the following results: 1) under conditions of uncertainty, the risk averse (preferring) firm will engage in less (more) legal and illegal trade than the risk neutral firm; and 2) the risk neutral firm will engage in less legal and illegal trade than it would under conditions of certainty.

The analysis begins by rewriting the FOC, eqs. 18 and 19 in the following manner,

$$E[U'(\pi)\cdot(P^f\cdot G_L + P^f\cdot(1-t))] = E[U'(\pi)]\cdot P^S, \tag{24}$$

$$E[U'(\pi)\cdot(P^f\cdot G_{S1}\cdot u)] = E[U'(\pi)]\cdot P^S. \tag{25}$$

Adopting Horowitz's (1970) alternative way of expressing the FOC, yields

$$P^f\cdot E[G_L+(1-t)] = P^S - \{P^f\cdot \mathrm{Cov}[U'(\pi),\ G_L+(1-t)] \ /\ E[U'(\pi)]\}, \tag{26}$$

and

$$P^f\cdot E[G_{S1}\cdot u] = P^S - \{P^f\cdot \mathrm{Cov}[U'(\pi),\ G_{S1}\cdot u] \ /\ E[U'(\pi)]\}. \tag{27}$$

From above it is clear that marginal revenue of the joint product is now a random variable. Furthermore, the marginal physical product (MPP) and marginal value product (MVP) of legal and illegal inputs are also random variables. Examining the covariance term in equations 26 and 27, it becomes clear that when $U''(\pi) = 0$, the covariance term is also equal to zero. The implication is that the risk neutral firm engages in joint product trade up to the point where the marginal cost of domestic production of exports (P^S) is equal to the E[MVP] of legal and illegal inputs. When $U''(\pi) \neq 0$, it will be demonstrated that the sign of the covariance term becomes dependent on the sign of $U''(\pi)$. With respect to the covariance term in equation 26, the derivatives

$$\partial U'(\pi)/\partial u = U''(\pi) \cdot P^f \cdot G_{S1} \cdot S \tag{28}$$

and

$$\partial [G_L+(1-t)]/\partial u = G_{LS1} \cdot S > 0 \tag{29}$$

confirms that the sign of the covariance term in equation 26 has the same sign as $U''(\pi)$, since the sign of equation 28 is dependent on $U''(\pi)$. However, the sign of the covariance term in equation 27 can not be ascertained. However, it can be demonstrated that when it is assumed that the elasticity of the marginal product curve ($\mathscr{E}$) for smuggling input (S^1) has an absolute value of less than one, then sign Cov = sign $U''(\pi)$:

$$\mathscr{E} = \partial G_{S1}/\partial S^1 \cdot S^1/G_{S1} = S^1 \cdot G_{S1S1}/G_{S1} > -1. \tag{30}$$

If equation (30) is true, then examining the derivatives of the two components of the covariance term in equation 27 with respect to u,

$$\partial [u \cdot G_{S1}]/\partial u = G_{S1} \cdot [1 + \mathscr{E}] > 0, \tag{31}$$

and

$$\partial U'(\pi)/\partial u = U''(\pi) \cdot P^f \cdot G_{S1} \cdot S, \tag{32}$$

verifies that sign Cov = sign $U''(\pi)$. That is, since the sign of equation 32 is dependent on $U''(\pi)$, and equation (31) is positive, sign Cov = sign $U''(\pi)$. The above results lead to the second proposition:

PROPOSITION II. ***When a firm engages in joint product trade under conditions of uncertainty, the risk averse (preferring) firm will engage in less (more) legal and illegal trade than the risk neutral firm.***

To establish the above proposition, the result of sign Cov = sign $U''(\pi)$ is applied to equations 26 and 27, and the following conditions are arrived at

$$P^f \cdot E[G_L] + P^f \cdot (1-t) \gtreqless P^s, \tag{33}$$

$$P^f \cdot E[G_{S1} \cdot u] \gtreqless P^s, \tag{34}$$

depending on whether $U''(\pi) \lesseqgtr 0$.

According to conditions 33 and 34, the marginal cost of producing the good to be used as legal and illegal inputs in joint product trade, (P^s), is less than, equal to, or greater than the expected marginal revenue of the tradeable, be it legal or illegal as $U'' \gtreqless 0$. This implies that the risk averse firm will engage in less legal and illegal trade than the risk neutral firm, and the risk neutral firm will engage in less legal and illegal trade than the risk preferring firm. This discussion establishes proposition II, given the conditions stated above.

Proposition II leads to the next issue: a comparison of the demand for legal and illegal inputs by the risk neutral firm to the firm operating in a certainty environment. Equations 12 and 13 represent the firm's FOC under certainty. Equations 26 and 27 represent the firm's FOC under uncertainty. Now, our attention will focus on the marginal product terms found in those two sets of equations. Under the assumption that the third derivative of the

smuggling production function exists, the marginal product terms are defined as functions of L and S under the FOC for the certainty case as:[6]

$$G_L(L,\hat{u}\cdot S) = (P^S/P^f)\cdot(1-t), \tag{12a}$$

$$\hat{u}\cdot G_{S1}(L,\hat{u}\cdot S) = P^S/P^f, \tag{13a}$$

and for the risk neutral firm operating under uncertainty,

$$E[G_L(L,u\cdot S)] = (P^S/P^f)\cdot(1-t) \tag{26a}$$

$$E[u\cdot G_{S1}(L,u\cdot S)] = P^S/P^f \tag{27a}$$

Equations 12a, 13a, 26a, and 27a leads to the third proposition,

PROPOSITION III: ***A risk neutral firm will engage in less legal and illegal trade, ceteris paribus, than it would under conditions of certainty.***

To establish the third proposition, it is assumed that the marginal product functions defined in equations 26a and 27a, are (themselves) concave functions. Concavity implies that the second order total differentials of the marginal product functions are negative semidefinite. The economic consequences of the concavity assumption are: 1) the elasticities of the marginal product curves are non-increasing functions of factor inputs,

$$\partial\eta/\partial L \leq 0, \quad \text{and } \partial\varepsilon/\partial S^1 \leq 0;^{7} \tag{35}$$

and 2) that legal and illegal inputs complement one another less and less as more of each input is employed in the production of a joint-product tradeable,

$$\partial G_{LS1}/\partial S^1 < 0 \text{ and } \partial G_{S1L}/\partial L < 0.^{8} \tag{36}$$

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[6] A certainty environment implies that the random variable u is replaced with its expected value, û. One must also remember in eqs. 12a, 13a, 26a, and 27a, that P^S represents the marginal cost of domestic production of the export good.

[7] The elasticity of the marginal product curve for legal trade in smuggling is defined as $\eta = L\cdot G_{LL} / G_L$.

[8] As noted by Ratti and Ullah, the concavity assumption imposed on the marginal product functions is consistent with many of the common forms of production functions used in economic analysis.

The concavity assumption allows a reapplication of Jensen's Inequality,

$$E[G_L(L,u\cdot S)] < G_L(L,\hat{u}\cdot S) \tag{37}$$

and

$$E[u\cdot G_{S1}(L,u\cdot S)] < \hat{u}\cdot G_{S1}(L,\hat{u}\cdot S). \tag{38}$$

The implication of inequality 37 for eqs. 12a and 26a is that the risk neutral firm will engage in less legal trade than if the firm was operating in a world of certainty. The implication of inequality 38 for eqs. 13a and 27a is that the risk neutral firm will engage in less illegal trade than if the firm was operating in a world of certainty. Thus, proposition III is established.

When smuggling incurs a real resource cost in the Pitt model, Pitt demonstrates that the welfare effect of smuggling is ambiguous. The introduction of uncertainty modifies Pitt's ambiguous welfare result derived under certainty. First, proposition I established that uncertainty reduces the ability of smugglers to transform ex-ante illegal goods into ex-post illegal goods, i.e., the melting ice effect is magnified. Next, proposition III established that under uncertainty, a risk neutral firm will engage in less legal and illegal trade, than in a certainty environment.

Propositions I & III indicate that total exports will decline when uncertainty is introduced. A decline in exports, implies a decline in imports and a shrinking of the country's trade triangle as compared to the certainty case presented by Pitt. Furthermore, the smuggler's transformation curve under uncertainty is inferior to the smuggler's transformation curve in a world of certainty. Therefore, the introduction of uncertainty causes a negative shift in the range of possible welfare levels presented by Pitt for his ambiguous welfare case, i.e., the trade triangle shrinks under uncertainty as compared to a world of certainty. In other words, the introduction of

uncertainty has a negative welfare effect regardless of whether the welfare effect of smuggling under certainty was positive or negative.[9] Examining this last statement in the context of proposition II, if it is assumed that smugglers are risk averse, then the negative shift in the range of possible welfare levels is even greater than for the risk neutral case.[10]

Sheik (1989) contends that in the smuggling literature, the risk associated with smuggling is modelled incorrectly. He argues that an implicit assumption in the literature is that smugglers are risk preferring. The foregoing analysis provides an example that contradicts his conclusion. However, under the assumption of risk preferring smugglers, smuggling under uncertainty would expand as compared to smuggling under certainty. The consequence of assuming risk loving smugglers in the above model would be a positive shift in the range of possible welfare levels.

With respect to Martin and Panagariya and other authors employing stochastic models of smuggling, their models are unable to make a distinction regarding the effect uncertainty has on smuggling behavior and welfare relative to smuggling in a world of certainty. The stochastic model of smuggling developed in this paper fills a void in the smuggling literature with respect to the effects uncertainty has on smuggling behavior and welfare as compared to a world of certainty.

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[9] In Pitt's paper, figure No. 2 and his discussion on page 453 demonstrates that welfare levels are bounded by U_S and U_P which indicates an ambiguous welfare result in his model. The introduction of uncertainty produces welfare levels bounded by $E[U_S]$ and $E[U_P]$. Comparing welfare ranges between certainty and uncertainty, the first three propositions indicate that $E[U_S] < U_S$ and $E[U_P] < U_P$.

[10] Sheikh (1989), notes that welfare under certainty is greater than welfare under uncertainty. However, he provides no formal proof.

VI. COMPARATIVE STATIC ANALYSIS I: CHANGES IN THE WORLD PRICE, THE MARGINAL COST OF DOMESTIC PRODUCTION, AND TAXES

In the next two sections it is assumed that the firm is risk neutral. The risk neutrality assumption is necessary in order to generate determinate results from the comparative static analysis. In order to analyze the affect of a change in the tax or world price of exports, the FOC, eqs. 18 and 19 are rewritten under the assumption of risk neutrality in the following manner,

$$Z_1 = E[P^f \cdot G_L + P^f \cdot (1-t) - P^S] = 0, \tag{39}$$

$$Z_2 = E[P^f \cdot G_{S1} \cdot u - P^S] = 0. \tag{40}$$

Invoking the implicit function theorem around the equilibrium values of L and S, and then taking the total differential of Z_1 and Z_2 leads to the fourth proposition:

PROPOSITION IV. ***An increase in the export tax rate, ceteris paribus, will reduce the risk neutral smuggling firm's demand for legal and illegal inputs. The decline in demand will reduce joint-product exports, which in turn will cause a decline in imports and welfare.***

To establish proposition IV all of the differentials except dL, dS, and dt are set to zero. It is now possible to derive the partial derivatives, $\partial L/\partial t$ and $\partial S/\partial t$,

$$\partial L/\partial t = P^f \cdot A_2 / DET < 0 \tag{41}$$

$$\partial S/\partial t = -P^f \cdot B_1 / DET < 0 \tag{42}$$

Equations 41 and 42 establish proposition IV for the risk neutral firm.[11] They indicate that an increase in the tax rate will reduce the equilibrium level of demand for legal and illegal inputs. This result implies that the consequence of a rise in the tax rate is a decline in joint-product exports. Which leads to a decline in imports and welfare.

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[11] The terms, A_1, A_2, B_1, and DET are defined by equations 20 through 23.

If all of the differentials except dL, dS, and dP^f are set to zero, then we have proposition V.

PROPOSITION V. *An increase in the world price of exports, ceteris paribus, will increase the smuggling firm's demand for legal and illegal inputs. The rise in demand will increase joint-product exports, which leads to an increase in the country's imports and welfare.*

To establish proposition V all of the differentials except dL, dS, and dP^f are set to zero. It is now possible to derive the partial derivatives, $\partial L/\partial P^f$ and $\partial S/\partial P^f$,

$$\partial L/\partial P^f = \{-E[G_L\cdot(1-t)]\cdot A_2 - (-E[u\cdot G_{S1}]\cdot B_1)\} / DET > 0 \tag{43}$$

$$\partial S/\partial P^f = \{-E[u\cdot G_{S1}]\cdot A_1 - (-E[G_L\cdot(1-t)]\cdot B_1)\} / DET > 0 \tag{44}$$

Equations 43 and 44 establishes proposition V for the risk neutral firm. They indicate that a rise in the world price for exports will increase the equilibrium level of demand for legal and illegal inputs. This result implies that the consequence of a rise in the world price for exports is an increase in joint-product exports, leading to a rise in imports and welfare.

If all of the differentials except dL, dS, and dP^S are set to zero, then we have proposition VI.

PROPOSITION VI. *An increase in the marginal cost for the domestic production of exports, ceteris paribus, will decrease the smuggling firm's demand for legal and illegal inputs. The decline in demand will decrease joint-product exports, which in turn will induce a decline in the country's imports and welfare.*

To establish proposition VI all of the differentials except dL, dS, and dP^S are set to zero. It is now possible to derive the partial derivatives, $\partial L/\partial P^S$ and $\partial S/\partial P^S$,

$$\partial L/\partial P^S = A_2 - B_1 / DET < 0 \tag{45}$$

$$\partial S/\partial P^S = A_1 - B_1 / DET < 0 \tag{46}$$

Equations 45 and 46 establish proposition VI for the risk neutral firm. Equations 45 and 46 demonstrate that a rise in the marginal cost of domestic production of the good to be exported will decrease the equilibrium level of demand for legal and illegal inputs. This result implies that the consequence of a rise in the marginal cost of domestic production is a decline in joint-product exports. Which leads to a decline in imports and welfare.

VII. COMPARATIVE STATIC ANALYSIS II: CHANGES IN THE LEVEL OF ENFORCEMENT AND UNCERTAINTY

A change in the level of enforcement or uncertainty implies a change in the moments of the random variable u, the smuggling success rate. Assume that the effect of an increase in anti-smuggling enforcement efficiency, implies a decline in the expected value of the smuggling success rate. This assumption leads to proposition VII,

PROPOSITION VII. ***An increase in enforcement activity against smuggling, ceteris paribus, will decrease the risk neutral firm's demand for legal and illegal inputs. The decline in demand will decrease the country's joint-product exports, leading to a decline in imports and welfare.***

To establish proposition VII a decline in the expected value of the smuggling success rate is analyzed by replacing u with $u^{*}= u+\theta$ in equations 39 and 40. Then, differentiating with respect to θ, and evaluating the resulting changes in the demand for legal and illegal inputs at $\theta=0$, produces equations 47 and 48.

A decline in θ has the effect of shifting the probability distribution of u to the left and decreasing the expected value of the smuggling success rate for each level of legal and illegal inputs, without altering the shape of the probability distribution. The result of this distribution preserving

shift in the mean is only determinate for the risk neutral case and is presented below in eqs. 47 and 48.

$$\partial L/\partial\theta = \{-E[P^f \cdot G_{LS1} \cdot S] \cdot A_2 - (-E[P^f \cdot (1+\mathcal{E})] \cdot B_1)\} \;/\; DET > 0 \tag{47}$$

$$\partial S/\partial\theta = \{-E[P^f \cdot (1+\mathcal{E})] \cdot A_1 - [-E[P^f \cdot G_{LS1} \cdot S] \cdot B_1\} \;/\; DET > 0 \tag{48}$$

Proposition VII is established under the assumption that $\mathcal{E}>-1$. This results in $\partial L/\partial\theta$ and $\partial S/\partial\theta$ both being positive. An implication of $\partial L/\partial\theta$ being positive, is that if enforcement activity against smuggling is increased, then export tax revenues will decline.

Next, the effect of a marginal increase in uncertainty is considered. To capture the effect of a marginal change in uncertainty, the distribution of u will undergo a mean preserving change in the dispersion of the distribution. The results developed below are only determinant in the risk neutral case. A modification of equations 39 and 40 is now undertaken by replacing u with $u^*=(\alpha \cdot u + \beta)$, where α is a shift parameter and β is a function of α with the following properties:

1) $\beta' = -E[u] = -\hat{u}$, and 2) $\beta(\alpha=1) = 0$. This transformation implies that $S^1 = (\alpha \cdot u + \beta) \cdot S$. This modification leads to the next proposition,

PROPOSITION VIII. *A mean preserving increase in uncertainty over how successful smuggling will be, ceteris paribus, will decrease the smuggling firm's demand for legal and illegal inputs. The decline in demand will decrease the country's joint-product exports, generating a decline in imports and welfare.*

To establish proposition VIII, it assumed the firm is risk neutral. Differentiating the transformed equations 39 and 40 with respect to α and evaluating $\partial L/\partial\alpha$ and $\partial S/\partial\alpha$ at $\alpha=1$, yields

$$\partial L/\partial\alpha = -P^f \cdot \{E[S \cdot G_{LS1} \cdot (u-\hat{u})] \cdot A_2 - E[G_{S1} \cdot (1+\mathcal{E}) \cdot (u-\hat{u})] \cdot B_1\} \;/\; DET \tag{49}$$

and

$$\partial S/\partial\alpha = -P^f \cdot \{E[G_{S1} \cdot (1+\mathcal{E}) \cdot (u-\hat{u})] \cdot A_1 - E[S \cdot G_{LS1} \cdot (u-\hat{u})] \cdot B_1\} \;/\; DET \tag{50}$$

The signs of the partial derivatives derived above can be determined by examining the following relationships:

$$E[S \cdot G_{LS1} \cdot (u-\hat{u})] = \mathrm{Cov}[S \cdot G_{LS1}, (u-\hat{u})] \tag{51}$$

and

$$E[G_{S1} \cdot (1+\mathscr{E}) \cdot (u-\hat{u})] = \mathrm{Cov}[G_{S1} \cdot (1+\mathscr{E}), (u-\hat{u})]. \tag{52}$$

By ascertaining the signs of the above covariance terms, the signs of the numerators in eqs. 49 and 50 can be determined.

Examining the derivatives of the three components of the covariance terms in eqs. 51 and 52 with respect to u yields,

$$\partial S \cdot G_{LS1}/\partial u < 0, \tag{53}$$

$$d(u-\hat{u})/\,du = 1 > 0, \tag{54}$$

and

$$\partial G_{S1} \cdot (1+\mathscr{E})/\partial u < 0. \tag{55}$$

Under the concavity assumption imposed on the smuggling production function and the marginal product functions discussed earlier, the signs of the covariance terms are negative. Negative covariance terms yield negative signs for the partial derivatives, $\partial L/\partial\alpha < 0$ and $\partial S/\partial\alpha < 0$, and establishes proposition VIII.

The comparative static analysis of a change in enforcement or uncertainty in this section was performed using the modelling techniques developed in the "competitive firm under uncertainty" literature. An analysis of the effect of increased uncertainty on smuggling behavior or increased enforcement's effect on the average rate of successful smuggling reveals that in a joint product model, legal and illegal trade decline. Therefore, welfare will decline as enforcement efficiency or uncertainty increase. The stochastic model developed in this paper makes a contribution by providing

insight into how enforcement and uncertainty effect the smuggling firm's input demand and joint-product (output supply) production decisions.

IX. A BRIBERY MODEL OF JOINT PRODUCT SMUGGLING

The real resource cost associated with smuggling was defined in equation 14 as being equal to S^1-S^*. Assume that bribes replace cloaking activities as the source of the "melting ice effect", the difference between S^1-S^*.[12] If bribery payments are consider to be just an income transfer, then the melting ice effect is rendered welfare neutral.

The substitution of bribery for cloaking activities in the certainty model presented earlier generates Pitt's strictly positive welfare effect for smuggling as compared to non-smuggling. However, the introduction of uncertainty in section IV, demonstrates that the mere presence of uncertainty will reduce the positive welfare effect of smuggling as compared to smuggling in a world of certainty. The implication of introducing uncertainty is that the smuggler's bribery cost increases as compared to smuggling in a world of certainty.

X. SUMMARY.

A stochastic model of joint-product smuggling was presented in this paper. The stochastic modeling approach used in this paper extends the microeconomic foundation of the smuggling literature by merging it with the literature on firm behavior under uncertainty.

The stochastic model developed in this paper extends the crime-theoretic approach to smuggling introduced by Martin and Panagariya. The model uses

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[12] Martin and Panagariya, and Sheikh both discuss the issue of bribery being used as a tool by smugglers.

enforcement as the activity which introduces uncertainty in the smuggler's production function. This allowed the modeling techniques developed in the "competitive firm under uncertainty" literature to be used to analyze smuggling under uncertainty for the first time.

The welfare and trade pattern implications of the preceding analysis provides marginal support for Bhagwati and Hansen's policy conclusion of the "less smuggling the better". The level of support is marginal because the very presence of uncertainty reduces welfare when compared to smuggling in a world of certainty.

REFERENCES

Batra, R.N., and Ullah, A., 1974, "Competitive Firm and the Theory of Input Demand Under Uncertainty", *Journal of Political Economy*, Vol. 82, No. 3, pp. 537-48.

Bhagwati, J. and B. Hansen, 1973, "A Theoretical Analysis of Smuggling", *Quarterly Journal of Economics* 87, pp. 172-187.

Fausti, S., 1992, "Smuggling and Parallel Markets for Exports", *The International Trade Journal* Vol. 6, No. 4, pp. 443-470.

Horowitz, I., 1970, Decision Making and Theory of the Firm, Holt Rinehart and Winston Publishing, Ny, Ny.

Martin, L. and Panagariya, A., 1984, "Smuggling, Trade, and Price Disparity: A Crime Theoretic Approach", *Journal of International Economics* 17, pp. 201-217.

Pitt, M., 1981, "Smuggling and Price Disparity", *Journal of International Economics* 11, pp. 447-458.

Rao, C.R., 1973, Linear Statistical Inference and its Applications, John Wiley Publishing, NY, NY.

Ratti, R.A., and Ullah, A., 1976, "Uncertainty in Production and the Competitive Firm", *Southern Economic Journal*, Vol. 42., pp. 703-10.

Samuelson, P., 1954, "The Transfer Problem and Transport Cost: Analysis of the Effects of Trade Impediments", *Economic Journal* 64, pp. 264-289

Sandmo, A., 1971, "On the Theory of the Competitive Firm Under Price Uncertainty", *American Economic Review*, Vol. 61, No. 1, pp. 65-73.

Schöler, K., 1989, "Risk and Illegal Trade", *Metroeconomica* 40, pp. 87-97.

Sheikh, M.A., 1989, "A Theory of Risk, Smuggling and Welfare", *World Development* 17, pp. 1931-1944.

Thursby, M., Jensen, R., and Thursby, J,. 1991, "Smuggling, Camouflaging, and Market Structure", *Quarterly Journal of Economics* 105, pp. 789-814.

MATHEMATICAL APPENDIX

Production Uncertainty, Enforcement and Smuggling: A Stochastic Model

1. Another way to state Eq. 16 would be:

$$E[G(L,S^1)] < G(L,E[S^1])$$

where $E[S^1] = \hat{u}\cdot S$

2. First order conditions, equations 18 and 19:

EQ. 18. $$\frac{\partial E[U(\pi)]}{\partial L} = E\left[\frac{dU(\pi)}{d\pi} \cdot \frac{\partial \pi}{\partial L}\right]$$

where $\frac{dU(\pi)}{d\pi} = U'(\pi)$

and $\frac{\partial \pi}{\partial L} = P^f \cdot G_L + P^f(1-t) \cdot L - P^s$

EQ. 19 $$\frac{\partial E[U(\pi)]}{\partial S} = E\left[\frac{dU(\pi)}{d\pi} \cdot \frac{\partial \pi}{\partial S}\right]$$

where $\frac{dU(\pi)}{d\pi} = U'(\pi)$, and

$$\frac{\partial \pi}{\partial S} = P^f \cdot \frac{\partial G}{\partial S^1}\,\frac{dS^1}{dS} - P^s\text{, or}$$

$$\frac{\partial \pi}{\partial S} = P^f G_{s_1} \cdot u - P^s$$

3. Second order conditions

$$EQ.\ 20\quad A_1 = \frac{\partial^2 E[U(\pi)]}{\partial L^2} = E\left[\frac{d^2U(\pi)}{d\pi^2} \cdot \frac{\partial\pi}{\partial L} + \frac{\partial^2\pi}{\partial L^2} \cdot \frac{dU(\pi)}{d\pi}\right] = A_1$$

$$A_1 = E[U''(\pi) \cdot (P^f \cdot G_L + P^f \cdot (1-t) - P^s)^2 + P^f \cdot G_{LL} \cdot U'(\pi)]$$

For equation 21

$$A_2 = \frac{\partial^2 E[U(\pi)]}{\partial S^2} = E\left[\frac{d^2U(\pi)}{d\pi^2}\ \frac{\partial\pi}{\partial S} + \frac{\partial^2\pi}{\partial S^2} \cdot \frac{dU(\pi)}{d\pi}\right]$$

$$A_2 = E\left[U''(\pi) \cdot (P^f \cdot u \cdot G_{s1} - P^s)^2 + P^f \cdot u^2 \cdot G_{s1s1} \cdot U'(\pi)\right]$$

Under the assumption of $U'' \leq 0$, A_1 and A_2 are both negative. When $U'' > 0$ it is assumed A_1 and A_2 are negative.

For equation 22

$$B_1 = \frac{\partial^2 E[U(\pi)]}{\partial L\partial S} = E\left[\frac{d^2U(\pi)}{d\pi^2}\ \frac{\partial\pi}{\partial L} + \frac{\partial^2\pi}{\partial L\partial S} \cdot \frac{dU(\pi)}{d\pi}\right]$$

$$B_1 = E\left[U''(\pi) \cdot (P^f \cdot G_L + P^f(1-t) - P^s) \cdot (P^f \cdot u \cdot G_{s1} - P^s) + U'(\pi) \cdot P^f \cdot u \cdot G_{Ls1}\right]$$

Taylor's Theorem states that

$$\frac{\partial^2 E[U(\pi)]}{\partial L\partial S} = \frac{\partial^2 E[U(\pi)]}{\partial S\partial L}$$

Imposing the second order conditions, we have equation 23.

4. Equations 24 and 25 are just equations 18 and 19 rearranged.
Equations 26 and 27 are written according to Horowitz, p. 364-367. Using the following definition E(xy) = E(x) · E(y) + COV(X,Y), equations 24 and 25 are transformed in equations 26 and 27. Remember that $S^1 = u \cdot S$

For equation 28

$$\frac{\partial U'(\pi)}{\partial u} = \frac{dU'(\pi)}{d\pi} \frac{\partial \pi}{\partial S^1} \frac{dS^1}{du}$$

$$\frac{dU'(\pi)}{d\pi} = U''(\pi)$$

$$\frac{\partial \pi}{\partial S^1} = P^f - G_{S1} \quad \wedge \quad \frac{dS^1}{du} = S$$

$$\frac{\partial U'(\pi)}{\partial u} = U''(\pi) \cdot P^f \cdot S \cdot G_{S1}$$

The sign of EQ. 28 is the same as $U''(\pi)$

For equation 29

$$\frac{\partial [G_L + (1-t)]}{\partial u} = \frac{\partial [G_L + (1-t)]}{\partial S^1} \cdot \frac{dS^1}{du}$$

$$\frac{\partial [G_L + (1-t)]}{\partial S^1} = G_{Ls1} \quad \wedge \quad \frac{dS^1}{du} = S$$

$$\frac{\partial [G_L + (1-t)]}{\partial u} = S \cdot G_{Ls1} > 0$$

The sign of Eq. 29 is positive, thus sign COV = Sign $U''(\pi)$ in Eq. 26.

For equation 31

$$\frac{\partial[u\cdot G_{s1}]}{\partial u} = G_{s1} + u\cdot\frac{\partial G_{s1}}{\partial S^1}\cdot\frac{dS^1}{du}$$

$$= G_{s1} + u\cdot G_{s1s1}\cdot S$$

$$= G_{s1}\cdot(1+\epsilon) > 0$$

where $u\cdot S = S_1$ and by EQ. 30, equation 31 is positive if $\epsilon > -1$.

Equation 32 is identical to Eq. 28. Therefore, the sign of Eq. 32 is the same as $U''(\pi)$. The sign of Eq. 31 is positive. Thus, SIGN COV = SIGN $U''(\pi)$ in Eq. 27.

5. The elasticity of the marginal product curve of legal trade is defined as

$$\eta = \frac{G_{LL}\cdot L}{G_L}$$

6. The assumptions imposed in eqs. 35 and 36 are consistent with the assumption that the marginal product functions are concave.

For equation 35

$$\frac{\partial\epsilon}{\partial S^1} = \frac{[S^1\cdot G_{s1s1s1} + G_{s1s1}]\cdot G_{s1} - S^1[G_{s1s1}]^2}{[G_{s1}]^2} < 0$$

Concavity assumption $\Rightarrow G_{s1s1s1} < 0$, $\wedge$ *it assures that* $\frac{\partial\epsilon}{\partial S^1} < 0$.

$$\frac{\partial\eta}{\partial L} = \frac{(G_{LLL}\cdot L + G_{LL})\cdot G_L - L\cdot(G_{LL})^2}{G_L^2} < 0$$

Concavity assumption $\Rightarrow G_{LLL} < 0 \wedge$ *it assures that* $\frac{\partial\eta}{\partial L} < 0$

For equation 36

$$\frac{\partial G_{Ls1}}{\partial S^1} = G_{Ls1s1} < 0 \quad \wedge \quad \frac{\partial G_{s1L}}{\partial L} = G_{s1LL} < 0$$

The Hessian matrices for the marginal product functions of legal and illegal trade are negative semidefinite if the conditions given in Eqs. 35 and 36 hold.

Legal Trade

$$H_L = \begin{bmatrix} G_{LLL} & G_{LLs1} \\ G_{Ls1L} & G_{Ls1s1} \end{bmatrix} \quad \wedge \quad |H_1| < 0 \quad |H_2| > 0$$

Illegal Trade

$$H_s = \begin{bmatrix} G_{s1s1s1} & G_{s1s1L} \\ G_{s1Ls1} & G_{s1LL} \end{bmatrix} \quad \wedge \quad |H_1| < 0 \quad |H_2| > 0$$

Total differentials of Eqs. 39 and 40. The total differentials for Z_1 and Z_2 are written under the assumption of risk neutrality. Thus, the FOC are no longer written in terms of utility.

$$dZ_1:\ E[P^f \cdot G_{LL} \cdot dL + P^f \cdot u \cdot G_{Ls1} \cdot dS] = -E[G_L \cdot dP^f + (1-t) \cdot dP^f + P^f \cdot G_{Ls1} \cdot S \cdot du - P^f \cdot dt - dP^s]$$

$$dZ_2:\ E[P^f \cdot G_{s1L} \cdot u \cdot dL + P^f \cdot u^2 \cdot G_{s1s1} \cdot dS] = -E[u \cdot G_{s1} \cdot dP^f + P^f \cdot G_{s1} \cdot du + P^f \cdot u \cdot G_{s1s1} \cdot S \cdot du - dP^s]$$

where du can be rewritten in the following manner

$$-E[P^f \cdot G_{s1} \cdot (1+\varepsilon)]\ dU$$

The Hessian matrix under risk neutrality

$$A = \begin{bmatrix} P^f \cdot G_{LL} & P^f \cdot u \cdot G_{Ls1} \\ P^f \cdot u \cdot G_{s1L} & P^f \cdot u^2 \cdot G_{s1s1} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ B_1 & A_2 \end{bmatrix}$$

$$|A| = DET\ A > 0$$

Comparative static analysis will employ Cramer's rule.

Eqs. 41 and 42, the effect of a change in the export tax rate on L, S.

$$A \cdot \begin{bmatrix} \frac{\partial L}{\partial t} \\ \frac{\partial S}{\partial t} \end{bmatrix} = \begin{bmatrix} P^f \\ 0 \end{bmatrix}$$

$$\frac{\partial L}{\partial t} = \frac{P^f \cdot A_2}{DET} < 0 \qquad \frac{\partial S}{\partial t} = \frac{-P^f \cdot B_1}{DET} < 0$$

Eqs. 43, 44, the effect of a change in the world price of exports on L, S.

$$A \cdot \begin{bmatrix} \frac{\partial L}{\partial P^f} \\ \frac{\partial S}{\partial P^f} \end{bmatrix} = \begin{bmatrix} -E[G_L + (1-t)] \\ -E[u \cdot G_{s1}] \end{bmatrix}$$

$$\frac{\partial L}{\partial P^f} = \frac{-E[G_L + (1-t)] \cdot A_2 - \{-E[u \cdot G_{s1}] \cdot B_1\}}{DET} > 0$$

$$\frac{\partial S}{\partial P^f} = \frac{A_1 \cdot [-E[u \cdot G_{s1}]] - B_1 [-E[G_L + (1-t)]]}{DET} > 0$$

Eqs. 45-46, the effect of a change in the marginal cost of domestic product for the good to be exported.

$$A \cdot \begin{bmatrix} \frac{\partial L}{\partial P^s} \\ \frac{\partial S}{\partial P^s} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{\partial L}{\partial P^s} = \frac{A_2 - B_1}{DET} < 0 \ \wedge \ \frac{\partial S}{\partial P^s} = \frac{A_1 - B_1}{DET} < 0$$

Eqs. 47-48, examine the effect of a distribution persevering shift of the mean, i.e., a change in enforcement effort.

$$A \cdot \begin{bmatrix} \frac{\partial L}{\partial \theta} \\ \frac{\partial S}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -E[P^f \cdot G_{Ls1} \cdot S] \\ -E[P^f \cdot G_{s1} \cdot (1+\varepsilon)] \end{bmatrix}$$

An increase in θ implies a decrease in enforcement efficiency. Thus, the firm's demand for legal and illegal inputs will increase. The comparative status are set forth in Equations 47, 48.

$$\frac{\partial L}{\partial \theta} = \frac{-E[P^f \cdot G_{Ls1} \cdot S] \cdot A_2 - B_1 \cdot [-E[P^f \cdot G_{s1} \cdot (1+\varepsilon)]}{DET} > 0$$

$$\frac{\partial S}{\partial \theta} = \frac{-E[P^f \cdot G_{s1} \cdot (1+\epsilon)] \cdot A_1 - B_1 \cdot [-E[P^f \cdot G_{Ls1} \cdot S]]}{DET} > 0$$

An increase in α implies an increase in uncertainty. An increase in uncertainty will decrease the firm's demand for legal and illegal inputs. The comparative static results are set forth in Equations 49, 50.

$$A \cdot \begin{bmatrix} \frac{\partial L}{\partial \alpha} \\ \frac{\partial S}{\partial \alpha} \end{bmatrix} = \begin{matrix} -P^f E[G_{Ls1} \cdot (u-\hat{u}) \cdot S] \\ -P^f E[G_{s1} \cdot [1+\epsilon] \cdot (u-\hat{u})] \end{matrix}$$

$$\frac{\partial L}{\partial \alpha} = \frac{-P^f E[G_{Ls1} \cdot (u-\hat{u}) \cdot S] \cdot A_2 - [-P^f E[G_{s1} \cdot (1+\epsilon) \cdot (u-\hat{u})] \cdot B_1}{DET}$$

$$\frac{\partial S}{\partial \alpha} \quad \frac{A_1 \cdot [-P^f E[G_{s1} \cdot (1+\epsilon) \cdot (u-\hat{u})] - [-P^f E[G_{Ls1} \cdot (u-\hat{u}) \cdot S] \cdot B_1}{DET}$$

The DET in Eqs. 49, 50, are positive thus, the signs of the partial derivatives $\partial L/\partial\alpha$, $\partial S/\partial\alpha$ are dependent on the signs of the numerators.

The signs of the partial derivatives derived above can be determined by examining the following relationships.

The sign of Eqs. 49 and 50 are dependent on the numerators, since the denominator is positive. The key to signing the numerators is the following relationship: $E(X \cdot Y) = E(X) \cdot E(Y) + COV(X,Y)$. Rewriting equations 49 and 50, we have,

49a. $E[G_{Ls1} \cdot S \cdot (u-\hat{u})] = E[S \cdot G_{Ls1}] \cdot E[u-\hat{u}] + COV[S \cdot G_{Ls1}, (u-\hat{u})]$

50a. $E[G_{s1} \cdot (1+\epsilon) \cdot (u-\hat{u}) = E[G_{s1} \cdot (1+\epsilon)] \cdot E[u-\hat{u}] + COV[G_{s1} \cdot (1+\epsilon), (u-\hat{u})]$

However, $E[u-\hat{u}] = 0$, so equations 49a and 50a reduce to eqs. 51 and 52.

51. $E[G_{Ls1} \cdot S \cdot (u-\hat{u})] = COV[S \cdot G_{Ls1}, (u-\hat{u})]$

52. $E[G_{s1} \cdot (1+\epsilon) \cdot (u-\hat{u})] = COV[G_{s1} \cdot (1+\epsilon), (u-\hat{u})]$

By ascertaining the signs of the covariance terms in equations 51 and 52, the signs of the numerators in equations 49 and 50 can be determined.

Examining the derivatives of the two components of the covariance terms with respect to u, will allow us to determine the signs of the covariance terms in eqs. 51 and 52.

Remembering that $S^1 = (\alpha \cdot u + B) \cdot S$, and

$\frac{dS^1}{du} = \alpha \cdot S \wedge S^1 = u^* \cdot S$, *we have*

$$\frac{\partial [S \cdot G_{Ls1}]}{\partial u} = S \cdot \left[\frac{\partial G_{Ls1}}{\partial S^1} \frac{dS_1}{du}\right]$$

$$= S \cdot G_{Ls1s1} \cdot \alpha \cdot S = \alpha \cdot S^2 \cdot G_{Ls1s1} < 0 \text{ , and}$$

$$\frac{d[u-\bar{u}]}{du} = 1 > 0, \text{ therefore } COV[S \cdot G_{Ls1}, (u-\bar{u})] < 0$$

The derivative of the other covariance term with respect to u is now analyzed below.

$$\frac{\partial [G_{s1} \cdot (1+\varepsilon)]}{\partial u} = \frac{\partial [G_{s1} + G_{s1s1} \cdot S^1]}{\partial u} =$$

$$\frac{\partial G_{s1}}{\partial S^1} \frac{dS^1}{du} + \frac{dS^1}{du} G_{s1s1} + \frac{\partial G_{s1s1}}{\partial S^1} \frac{dS^1}{du} S^1 =$$

$$G_{s1s1} \cdot \alpha \cdot S + \alpha \cdot S \cdot G_{s1s1} + G_{s1s1s1} \cdot \alpha \cdot S \cdot S^1 < 0 \text{, and}$$

$$\frac{d(u-\bar{u})}{du} = 1 > 0, \text{ therefore } COV[Gs1 \cdot (1+\varepsilon), (u-\bar{u})] < 0 .$$

The above results demonstrate that the numerators in eqs. 49 and 50 are negative and thus we have the following results,

$$\frac{\partial S}{\partial \alpha} < 0 \text{ and } \frac{\partial L}{\partial \alpha} < 0 .$$