On statistical estimates of the Inverted Kumaraswamy Distribution under adaptive progressive type-I hybrid censoring

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- 2. Progressive censoring scheme
- 3. MLE
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Inverted Kumaraswamy distribution

Probability density function

$$f(x;\alpha,\theta) = \alpha\theta(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\theta-1}, \; x > 0; \alpha,\theta > 0,$$

Cumulative distribution function

$$F(x;\alpha,\theta) = (1-(1+x)^{-\alpha})^{\theta}, \; x > 0; \alpha,\theta > 0,$$

Reliability function

$$R(x) = P(T > t) = 1 - F(t) = 1 - (1 - (1 + x)^{-\alpha})^{\theta} \;, x > 0$$

Hazard rate function

$$h(x) = \frac{f(x)}{R(x)} = \frac{\alpha\theta(1 + x)^{-(\alpha+1)}(1 - (1 + x)^{-\alpha})^{\theta-1}}{1 - (1 - (1 + x)^{-\alpha})^{\theta}} \;, x > 0;\ \alpha, \theta > 0$$

Quantile function

$$q(p) = \left[\left(1 - (p)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}} - 1\right], 0 < \mathrm{p} < 1.$$

Adaptive progressively type-I hybrid censoring

Progressive type II censoring scheme

- Given a positive integer $m \leq n$ and R_i for $i = 1,2,3,\dots,m$ such that $n = \sum_{i=1}^{m}(R_i+1)$
- Let n items be put on the life test at the same initial time, $t_0 = 0$. At the ith failure time , $X_{i:n}$, randomly select R_i items from $n - R_{i-1} - i$ surviving items, where $i = 1,2,3,\dots,m$ and $R_0 = 0$. Therefore, $X_{i:n} < X_{i+1:n}$ for $i = 1,2,3,\dots m-1$.

Adaptive progressive type I hybrid censoring scheme

Given $\tau > 0$ and progressive type II censoring scheme $\{R_j, j = 1,2,3, \ldots, m\}$ such that $n = \sum_{i=1}^{m}(R_i+1)$

- Implement progressive type II censoring scheme with $\{R_j, j = 1,2,3, \ldots, m\}$ and must terminate at time τ.
- Let D be the number of failed items just right afore τ.
- If the mth failure time, $X_{m:n}$ is obtained before τ, the life test experiment will continue to observe failures without withdrawing survival items until τ. All survival items $R_D^* = n - D - \sum_{i=1}^{D} R_i$ will be removed at time τ . Here, $R_m = R_{m+1} = \cdots = R_D = 0$ when $m \leq D$. Hence, the adaptive progressive type I censoring scheme is $R_1, R_2, R_3, \ldots, R_D$.

Likelihood Function

- Let $\Phi = \{x_{j:n}, j = 1,2,3, \dots D\}$ be the adaptive progressively type-I hybrid censored sample that was collected by using the progressive type-II censoring scheme, $\{R_{j,}\, j = 1,2,3, \dots m\}$, and τ.
- When the sample is collected from InKum (α, θ), the likelihood function can be represented as follows:

$$L(\alpha, \theta; \Phi) = (1 - (1 - (1 + \tau)^{-\alpha})^{\theta})^{n} \quad \text{for D} = 0;$$

Otherwise, the likelihood function can be represented as:

$$L(\alpha, \theta; \Phi) = C_D \alpha^D \theta^D \prod_{j=1}^{D} (1 + x_{j:n})^{-(\alpha+1)} (1 - (1 + x_{j:n})^{-\alpha})^{\theta - 1} \prod_{j=1}^{D} (1 - (1 - (1 + x_{j:n})^{-\alpha})^{\theta})^{R_j} (1 - (1 - (1 + \tau)^{-\alpha})^{\theta})^{R_D^*}$$

where $C_D = \prod_{i=1}^{D} \Upsilon_i$, $\Upsilon_i = \sum_{k=i}^{m} (R_k + 1) = n - \sum_{k=1}^{i-1} (R_k + 1)$, $\sum_{k=1}^{0} (R_k + 1) \equiv 0$, and $R_D^* = \text{n} - \text{D} - \sum_{j=1}^{D} R_j$.

- When D > 0, the log likelihood can be presented as:

$$lL(\alpha, \theta; \Phi) = \log(C_D) + D\log(\theta) + \log(\alpha) + (\theta - 1)\sum_{j=1}^{D}\log(1 - \left(1 + x_{j:n}\right)^{-\alpha}\right) - (\alpha + 1)\sum_{j=1}^{D}\log\left(1 +$$

Maximum Likelihood Estimator (MLE)

- By calculating the first partial derivatives of $lL(\alpha,\theta)$ with respective to α and θ, respectively, then the solution to this equation: $\left(\frac{\partial}{\partial\alpha} lL(\alpha,\theta), \frac{\partial}{\partial\theta} lL(\alpha,\theta)\right) = (0,0)$, is the MLE of (α,θ), which is labeled by $(\hat{\alpha},\hat{\theta})$

- Let $\Theta = (\alpha,\theta)$ and $\widehat{\Theta} = (\hat{\alpha},\hat{\theta})$, under some regular conditions, $\sqrt{n}(\widehat{\Theta} - \Theta) \rightarrow N(\mathbf{0}, \boldsymbol{I}^{-1}(\Theta))$, then Fisher information matrix defined by:

$$I(\Theta) = \frac{-1}{n}\begin{bmatrix} E(\frac{\partial^2 lL(\Theta)}{\partial\alpha^2}) & E(\frac{\partial^2 lL(\Theta)}{\partial\alpha\partial\lambda}) \\ E(\frac{\partial^2 lL(\Theta)}{\partial\lambda\partial\alpha}) & E(\frac{\partial^2 lL(\Theta)}{\partial\lambda^2}) \end{bmatrix} = -\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

- Let $\eta(\alpha, \theta)$ be a function of α and θ. For example, $\eta(\alpha, \theta) = \alpha, \eta(\alpha, \theta) = \theta, \eta(\alpha, \theta) = R(t),$
- $\eta(\alpha, \theta) = h(t), \eta(\alpha, \theta) = q(p)$. The MLE of η is $\hat{\eta}$ = $\eta(\hat{\alpha}, \hat{\theta})$.

- Using delta method, $\frac{\hat{\eta} - \eta(\Theta)}{\sigma_{\hat{\eta}}} \longrightarrow$ N(0,1), where $\sigma_{\hat{\eta}}^2 = \nabla\eta(\Theta)^T I^{-1}(\Theta) \nabla\eta(\Theta)$, we can get the 1- α confidence interval of $\eta(\alpha, \theta)$:

$$\hat{\eta} - z_{\alpha/2}\sigma_{\hat{\eta}} < \eta(\alpha, \theta) < \hat{\eta} + z_{\alpha/2}\sigma_{\hat{\eta}}$$

Simulation Procedure

Given $\alpha, \theta, m\ (\leq n), \tau$ and a progressive type II censoring scheme $\{R_j, j = 1,2,3, \ldots, m\}$ and
Let $\eta(\alpha, \theta)$ be any aforementioned function of α, θ

- Step 1: generate adaptive progressively type I hybrid censored sample $\Phi = \{x_{j:n}, j = 1,2,3, \ldots, D\}$ and resulting the adaptive progressive type-II censoring scheme, $\{R_j, j = 1,2,3, \ldots, D, D^*\}$.
- Step 2: Find the MLE of $\eta(\alpha, \theta)$ based on sample obtained from Step 1 and label the MLE by $\hat{\eta} = \eta(\hat{\alpha}, \hat{\theta})$, where $\hat{\alpha}$ and $\hat{\theta}$ are the MLEs of α and θ, respectively.
- Step 3: Repeat Step 1 and Step 2 for 10,000 times and label all MLEs by $\hat{\eta}_j$, $\mathrm{j} = 1,2,3, \ldots, 10000$.
- Step 4: Calculate the simulated MSE that is defined as $\frac{1}{10000}\sum_{j=1}^{10000}\left(\hat{\eta}_j - \eta\right)^2$ and bias that is defined as $\frac{1}{10000}\sum_{j=1}^{10000}\hat{\eta}_j - \eta$

Confidence interval through asymptotic Normal distribution

- For each MLE obtained from simulation procedure, an asymptotic $1-\alpha$ confidence is obtained through the following formula

$$\hat{\eta}_j - z_{\alpha/2}\hat{\sigma}_{\hat{\eta}_j} < \eta < \hat{\eta}_j + z_{\alpha/2}\hat{\sigma}_{\hat{\eta}_j} \text{ for } j = 1,2,3,\ldots,10000$$

- Find the relative frequency of 10,000 confidence intervals that cover the true η as the calculated the probability of coverage.

Confidence interval through Bootstrap sampling Procedure

- For given each MLE $\hat{\alpha}_j$ and $\hat{\theta}_j$ generated from the simulation procedure $j = 1,2,3,\dots,10000$
- Step 1: generate adaptive progressively type I hybrid censored sample $\Phi = \left\{x^*_{j,n}, j = 1,2,3,\dots,D\right\}$ from InKum$(\hat{\alpha}_j, \hat{\theta}_j)$ and resulting the adaptive progressive type-II censoring scheme, $\{R^*_j, j = 1,2,3,\dots,D,D^*\}$.
- Step 2: Find MLE for $\hat{\eta}^*$ based on bootstrap sample obtained from Step 1.
- Step 3: Repeat Step 1 and Step 2 for 10000 times and label 10000 MLEs by $\hat{\eta}^*_k$, for $k = 1,2,\dots,10000$.
- Step 4: Find empirical distribution based on sample from Step 3 and highest density $1-\alpha$ confidence interval by R function hdi in library HDinterval.
- Step 5: Find the relative frequency of 10,000 confidence intervals that cover the true η as the calculated the probability of coverage.

Tables

- Delta Method with Asymptotic Normal Distribution
- Bootstrap Method

Delta Method with Asymptotic Normal Distribution

Sch I for Inverted Kumaraswamy distribution $(\alpha, \theta) = (2.5, 4.5)$

Simulated MSE (Bias) for point estimator of Inverted Kumaraswamy distribution $(\alpha, \theta) = (2.5, 4.5), \tau = 1.5$.

Sch I		Distribution Parameters		percentiles			$\tau = 1.5$	
m	n	α	θ	$q_{0.25}$	$q_{0.5}$	$q_{0.75}$	R_τ	h_τ
20	60	0.19369 (0.084249)	2.72015 (0.48249)	0.0060135 (0.009848)	0.019437 (0.01089)	0.11337 (0.02067)	0.003649 (-0.002269)	0.027005 (0.030537)
20	80	0.14066 (0.06127)	1.76807 (0.341157)	0.0004421 (0.006649)	0.014114 (0.007286)	0.08095 (0.01431)	0.0027057 (-0.001908)	0.019544 (0.022433)
20	120	0.04107 (0.03255)	0.002412 (0.007186)	0.000214 (0,00260)	0.004958 (0.00892)	0.22635 (0.07369)	0.001712 (-0.000746)	0.005333 (0.011053)

Sch I. For Inverted Kumaraswamy distribution $(\alpha, \theta) = (2.5, 4.5)$

Simulated 95% CI (CP) of Inverted Kumaraswamy distribution $(\alpha, \theta) = (2.5, 4.5), \tau = 1.5$,

Sch I		α		θ		R_τ		h_τ	
m	n	AL	AU	AL	AU	AL	AU	AL	AU
20,	60	1.75106	3.4174	2.16447	7.8005	0.000	0.76622	0.5417	1.16369
		CP (0.9502)		CP(0.961)		CP (1.00)		CP(0.9477)	
20	80	1.84416	3.2784	2.49538	7.1869	0.04488	0.71869	0.5772	1.11199
		CP(0.9477)		CP(0.9572)		CP(0.9572)		CP(0.949)	
20	120	1.95740	3.12121	2.87026	6.5691	0.10755	0.6528	0.6198	1.05329
		CP(0.9477)		CP(0.9532)		CP(1.00)		CP(0.9481)	

Sch I	$q_{0.25}$		$q_{0.5}$		$q_{0.75}$	
m n	AL	AU	AL	AU	AL	AU
20 60	0.56209	0.85891	0.92201	1.4568	1.42408	2.7024
	CP(0.9466)		CP(0.9481)		CP(0.9308)	
20 80	0.57887	0.83572	0.95548	1.41614	1.50971	2.6041
	CP(0.9453)		CP(0.947)		CP(0.937)	
20 120	0.60027	0.81007	0.99641	1.37121	1.6106	2.4962
	CP(0.9457)		CP(0.9475)		CP(0.9424)	

Sch I for Inverted Kumaraswamy distribution $(\alpha, \theta) = (2.5, 4.5)$

Simulated 95% CI (CP) of Inverted Kumaraswamy distribution $(\alpha, \theta) = (2.5, 4.5), \tau = 1.5,$

Sch I		α		θ		R_τ		h_τ	
m	n	AL	AU	AL	AU	AL	AU	AL	AU
20	60	1.7709	3.4667	2.6235	8.6279	0.2667	0.5036	0.5488	1.1779
		CP (0.9996)		CP(0.9933)		CP (0.9987)		CP(0.999)	
20	80	1.8635	3.3116	2.8285	7.7323	0.2815	0.4860	0.5834	1.1214
		CP(0.9999)		CP(0.9918)		CP(0.9999)		CP(0.9999)	
20	120	1.9776	3.1468	3.1027	6.9247	0.2995	0.4662	0.6259	1.0606
		CP(0.9999)		CP(0.9908)		CP(0.9999)		CP(0.9999)	

Sch I	$q_{0.25}$		$q_{0.5}$		$q_{0.75}$	
m n	AL	AU	AL	AU	AL	AU
20 60	0.5750	0.8778	0.9440	1.4940	1.4934	2.7962
	CP(0.9561)		CP(0.998)		CP(0.9999)	
20 80	0.5888	0.8482	0.9723	1.4370	1.5606	2.6504
	CP(0.9626)		CP(0.9993)		CP(0.9999)	
20 120	0.6080	0.8185	1.0084	1.3829	1.6428	2.5174
	CP(0.9768)		CP(0.9999)		CP(0.9999)	

Conclusion

- It is found that in this study, the MSE and bias are acceptable.
- The coverage probabilities obtained by delta method through asymptotic normal distribution are closer to nominal one(0.95) than those obtained by bootstrap method.