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Some 2-Color Rado Numbers For A Linear Equation With A Negative Constant

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THESIS ACCEPTANCE PAGE

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This thesis is approved as a creditable and independent investigation by a candidate for the master's degree and is acceptable for meeting the thesis requirements for this degree. Acceptance of this does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

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Abstract

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An r-coloring is a function Δ that assigns a color to each natural number from 1 to some number n using colors $0, 1, \ldots, r-1$. A monochromatic solution (in Δ) to an equation L with m variables is an ordered m-tuple $(x_1, x_2, \ldots, x_m)$ where $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m)$ and $(x_1, x_2, \ldots, x_{m-1}, x_m)$ solves L. Given a linear equation L and $t \in \mathbb{N}$, the *t-color Rado number* for L is the least integer n (if it exists) such that every $\Delta : [1, n] \rightarrow [0, t-1]$ admits a monochromatic solution to L. If no such integer exists, the t-color Rado number for L is infinite. We prove the following two theorems.

Theorem. *The two-color Rado number for the equation*

$$x_1 + x_2 + x_3 + c = x_4$$

with $c < 0$ is

$$R(4,c) = \begin{cases} -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1 & m = 4,\ c \textit{ even} \\ \\ \infty & m = 4,\ c \textit{ odd} \end{cases}$$

Theorem. *The two-color Rado number for the equation*

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

with $c < 0$ *is*

$$R(5,c) = \begin{cases} -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil + 1 & m = 5,\ c \equiv 0 \ \textit{(mod 3)} \\ 7 & m = 5,\ c = -2 \\ -\frac{c+2}{3} + 2 & m = 5,\ -11 \leq c \leq -5 \ \textit{and}\ c \equiv 1 \ \textit{(mod 3)} \\ -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2 & m = 5,\ c < -11 \ \textit{and}\ c \equiv 1 \ \textit{(mod 3)} \\ 13 & m = 5,\ c = -1 \\ 5 & m = 5,\ c = -4 \\ -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2 & m = 5,\ c < -4 \ \textit{and}\ c \equiv 2 \ \textit{(mod 3)} \end{cases}$$

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Chapter 1

Introduction

Throughout this paper, we use $[a, b]$ to denote the set $\{a, a+1, \ldots, b-1, b\}$ where a and b are integers with $a \leq b$. A *coloring* is a function that assigns a color to each natural number from 1 to some number n. We use $\Delta : [1, n] \to [0, r-1]$ to denote an r-coloring of the natural numbers from 1 to n using colors $0, 1, \ldots, r-1$. A *monochromatic solution* (in Δ) to an equation L with m variables is an ordered m-tuple $(x_1, x_2, \ldots, x_m)$ where $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m)$ and $x_1, x_2, \ldots, x_{m-1}, x_m$ solves L.

We will start by introducing a major theorem from combinatorics that was proven by Issai Schur in 1916 [4].

Theorem 1 (Schur's Theorem). *For every finite $t \geq 2$, there exists a least integer $n = S(t)$ such that every coloring $\Delta : [1, n] \to [0, t-1]$ admits a monochromatic solution to the equation $x_1 + x_2 = x_3$.*

$S(t)$ is called the *t-color Schur number* and the equation $x_1 + x_2 = x_3$ is called *Schur's equation.*

Schur's theorem is part of a branch of combinatorics called Ramsey Theory. Ramsey Theory has two major areas: coloring the natural numbers, and coloring the edges of a graph. A *graph* is a set of vertices where some edges exist between the vertices. A *complete graph* is a graph where there is an edge between each pair of vertices. The following theorem, proven in 1930 by Frank Ramsey[5], considers coloring the edges of a graph using t colors.

Theorem 2 (Ramsey's Theorem)**.** *For every integer $t \geq 2$ and every $s_i \in \mathbb{N}$ with $i \in [1,t]$, there exists a least integer $n = R(s_1, s_2, \ldots, s_t)$ such that for every t-coloring of the edges of a complete graph on n vertices there exists a complete graph on s_i vertices monochromatic in color i for some i.*

In the next section, we will prove Schur's theorem using Ramsey's theorem.

Richard Rado, one of Schur's students, worked on a variation of Schur's problem by making a modification to the equation. Because of this we have the following definition.

Definition 1 (Rado Number)**.** Given a linear equation L and $t \in \mathbb{N}$, the *t-color Rado number* for L is the least integer n (if it exists) such that every coloring $\Delta : [1,n] \to [0, t-1]$ admits a monochromatic solution to L. If no such integer exists, the *t-color Rado number* for L is infinite.

Note that Schur numbers always exist, but Rado numbers do not always exist. In other words, Schur numbers are always finite while some Rado numbers may be infinite.

Definition 2 ($L(m,c)$, $R(m,c)$)**.** Let $L(m,c)$ denote the equation $x_1 + x_2 + \cdots + x_{m-1} + c = x_m$ and let $R(m,c)$ denote the 2-color Rado number for $L(m,c)$.

In order to visualize a coloring, we use the following notation to denote that 1 is colored with color a, 2 is colored with color b, 3 is colored with color c, and so on.

$$\frac{a}{1}\ \frac{b}{2}\ \frac{c}{3}\cdots$$

In 1982, Beutelspacher and Brestovansky [1] considered one modification of Schur's equation by adding more variables to get $L(m,0) : x_1 + x_2 + \cdots + x_{m-1} = x_m$. They found the following result, which we prove in the following section.

Theorem 3. *The 2-color Rado number for $L(m,0)$ with $m \geq 3$ is*

$$R(m,0) = m^2 - m - 1$$

Before stating the next result, we need the following definition.

Definition 3 (Ceiling)**.** The *ceiling* of x, denoted $\lceil x \rceil$, is the least integer in the interval $[x, x+1)$.

Another modification of Schur's equation was investigated by Burr, Loo and Schaal [2]. For the equation $L(3,c) : x_1 + x_2 + c = x_3$, they found the following result.

Theorem 4. *The 2-color Rado number for L(3,c) is*

$$R(3,c) = \begin{cases} 4c+5 & c \geq 0 \\ \lceil \frac{-4c+1}{5} \rceil & c < 0 \end{cases}$$

We prove this result for $c \geq 0$ in the next section.

Schaal combined the above two modifications of Schur's equation and considered the equation $x_1 + x_2 + \cdots + x_{m-1} + c = x_m$ for non-negative values of c [3]. He proved the following theorem.

Theorem 5. *For $m \geq 3$ and $c \geq 0$,*

$$R(m,c) = \begin{cases} \infty & m \text{ even, } c \text{ odd} \\ m^2 - m - 1 + c(m+1) & \text{otherwise} \end{cases}$$

We wish to continue this problem for $c < 0$. Note that $m = 3$ with $c < 0$ is done (Theorem 4). We will give a result for $R(4,c)$ and $R(5,c)$ with $c < 0$.

Chapter 2

Background

We will prove several of the theorems stated in the previous section. First, we will prove Schur's theorem using Ramsey's theorem. Although Schur proved his theorem 14 years before Ramsey's theorem was proven, the proof of Schur's theorem is simplest with the use of Ramsey's theorem.

Theorem 1 (Schur's Theorem)**.** *For every finite $t \geq 2$, there exists a least integer $n = S(t)$ such that every coloring $\Delta : [1, n] \to [0, t-1]$ admits a monochromatic solution to the equation $x_1 + x_2 = x_3$.*

Proof of Theorem 1. Let $t \geq 2$ be finite. Then let $n = R(3, 3, \ldots, 3) - 1$ from Ramsey's Theorem (Theorem 2) with t colors. Let $\Delta : [1, n] \to [0, t-1]$. Let K be a complete graph on $n+1$ vertices where the vertices are labeled $1, \ldots, n, n+1$. Color the edge xy of K $\Delta(|x-y|)$ for all vertices x and y. By Ramsey's Theorem, K contains a complete graph on 3 vertices that is monochromatic. Denote these vertices i, j, and k with $i > j > k$. Then $\Delta(i-j) = \Delta(j-k) = \Delta(i-k)$. Also $(i-j)+(j-k) = (i-k)$. Let $x_1 = i - j$, $x_2 = j - k$, and $x_3 = i - k$. So Δ admits monochromatic solution $x_1 + x_2 = x_3$. □

Next, we prove Beutelspacher and Brestovansky's results for the equation $x_1 + x_2 + \cdots + x_{m-1} = x_m$.

Theorem 3. *The 2-color Rado number for $L(m,0)$ with $m \geq 3$ is*

$$R(m,0) = m^2 - m - 1$$

Proof of Theorem 3. Consider the equation $L(m,0) : x_1 + x_2 + \cdots + x_{m-1} = x_m$. To show that $R(m,0) = m^2 - m - 1$, we first demonstrate a coloring of length $m^2 - m - 2$ with no monochromatic solution to $L(m,0)$. This gives a lower bound: $R(m,0) \geq m^2 - m - 1$. Then we show that every coloring of length $m^2 - m - 1$ admits a monochromatic solution to $R(m,0)$. This gives an upper bound: $R(m,0) \leq m^2 - m - 1$.

Lower Bound:

Consider the coloring Δ which colors as follows:

$$\frac{0}{1} \;\cdots\; \frac{0}{m-2} \;\; \frac{1}{m-1} \;\cdots\; \frac{1}{m^2-2m} \;\; \frac{0}{m^2-2m+1} \;\cdots\; \frac{0}{m^2-m-2}$$

Let $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m) = 1$. Then

$$\begin{aligned} x_1 + x_2 + \cdots + x_{m-1} &\geq (m-1) + (m-1) + \cdots + (m-1) \\ &= (m-1) \cdot (m-1) \\ &= m^2 - 2m + 1 \\ &> m^2 - 2m \\ &\geq x_m \end{aligned}$$

So $(x_1, x_2, \ldots, x_{m-1}, x_m)$ is not a solution to $L(m, 0)$.

Let $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m) = 0$. If $x_1, x_2, \ldots, x_{m-1} \in \{1, 2, \ldots, m-2\}$, then

$$x_1 + x_2 + \cdots + x_{m-1} \geq 1 + 1 + \cdots + 1 = 1 \cdot (m-1) = m - 1$$

and

$$\begin{aligned} x_1 + x_2 + \cdots + x_{m-1} &\leq (m-2) + (m-2) + \cdots + (m-2) \\ &= (m-2) \cdot (m-1) \\ &= m^2 - 3m + 2 \\ &= m^2 - 2m + (-m + 2) \\ &\leq m^2 - 2m \end{aligned}$$

since $m \geq 3$. Then $m - 1 \leq x_1 + x_2 + \cdots + x_{m-1} \leq m^2 - 2m$, but $x_m \notin [m-1, m^2 - 2m]$ so $(x_1, x_2, \ldots, x_{m-1}, x_m)$ is not a solution to $L(m, 0)$. Otherwise, $\exists x_i$ (where $1 \leq i \leq m-1$) with $x_i \geq m^2 - 2m + 1$. Then

$$\begin{aligned} x_1 + x_2 + \cdots + x_{m-1} &\geq (m^2 - 2m + 1) + 1 + \cdots + 1 \\ &= (m^2 - 2m + 1) + 1 \cdot (m-2) \\ &= m^2 - m - 1 \\ &> x_m \end{aligned}$$

So $(x_1, x_2, \ldots, x_{m-1}, x_m)$ is not a solution to $L(m, 0)$.

Therefore Δ does not admit a monochromatic solution to $L(m, 0)$. We have shown that there exists a coloring $\Delta : [1, m^2 - m - 2] \to [0, 1]$ which does not admit a monochromatic solution to $L(m, 0)$. Therefore, $R(m, 0) \geq m^2 - m - 1$.

Upper Bound:

Let $\Delta : [1, m^2 - m - 1] \to [0, 1]$ be any coloring. We show that Δ admits a monochromatic solution to $L(m, 0)$. Without loss of generality, assume $\Delta(1) = 0$.

If $\Delta(m - 1) = 0$, then $(1, 1, \ldots, 1, m - 1)$ is a monochromatic solution to $L(m, 0)$. Otherwise $\Delta(m - 1) = 1$.

If $\Delta(m^2 - 2m + 1) = 1$, then $(m - 1, m - 1, \ldots, m - 1, m^2 - 2m + 1)$ is a monochromatic solution to $L(m, 0)$. Otherwise $\Delta(m^2 - 2m + 1) = 0$.

If $\Delta(m) = 0$, then $(1, m, m, \ldots, m, m^2 - 2m + 1)$ is a monochromatic solution to $L(m, 0)$. Otherwise $\Delta(m) = 1$.

Then if $\Delta(m^2 - m - 1) = 1$, $(m - 1, m, m, \ldots, m, m^2 - m - 1)$ is a monochromatic solution to $L(m, 0)$. Also, if $\Delta(m^2 - m - 1) = 0$ then $(1, 1, \ldots, 1, m^2 - 2m + 1, m^2 - m - 1)$ is a monochromatic solution to $L(m, 0)$. So Δ must admit a monochromatic solution to $L(m, 0)$. Then $R(m, 0) \leq m^2 - m - 1$. Therefore $R(m, 0) = m^2 - m - 1$. $\square$

We prove Burr, Loo, and Schaal's result for the equation $x_1 + x_2 + c = x_3$ with $c \geq 0$.

Theorem 4. *The 2-color Rado number for L(3,c) is*

$$R(3,c) = \begin{cases} 4c+5 & c \geq 0 \\ \\ \left\lceil \frac{-4c+1}{5} \right\rceil & c < 0 \end{cases}$$

Proof of Theorem 4 for $c \geq 0$. Consider the equation $L(3,c) : x_1 + x_2 + c = x_3$

Lower Bound:

Consider the coloring Δ which colors as follows:

$$\frac{0}{1} \;\cdots\; \frac{0}{c+1} \;\; \frac{1}{c+2} \;\cdots\; \frac{1}{3c+3} \;\; \frac{0}{3c+4} \;\cdots\; \frac{0}{4c+4}$$

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = 0$. If $x_1, x_2 \in [1, c+1]$, then

$$x_1 + x_2 + c \geq 1 + 1 + c = c + 2$$

and

$$x_1 + x_2 + c \leq (c+1) + (c+1) + c = 3c + 2$$

But $x_3 \notin [c+2, 3c+2]$ so (x_1, x_2, x_3) is not a monochromatic solution.

If $\exists i \in \{1,2\}$ such that $x_i \geq 3c+4$, then

$$\begin{aligned} x_1 + x_2 + c &\geq 1 + (3c+4) + c \\ &= 4c+5 \\ &> 4c+4 \\ &\geq x_3 \end{aligned}$$

So Δ does not admit a solution monochromatic in 0.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = 1$. Then

$$\begin{aligned} x_1 + x_2 + c &\geq (c+2) + (c+2) + c \\ &= 3c+4 \\ &> 3c+3 \\ &\geq x_3 \end{aligned}$$

So Δ does not admit a solution monochromatic in 1. Therefore Δ does not admit a monochromatic solution to $x_1 + x_2 + c = x_3$. So $R(3,c) \geq 4c+5$ for all $c \geq 0$.

<u>Upper Bound:</u>

Let $\Delta : [1, 4c+5] \to [0,1]$ be any coloring. We show that Δ admits a monochromatic solution to $L(3,c)$. Without loss of generality, assume $\Delta(1) = 0$.

If $\Delta(c+2) = 0$, then $(1, 1, c+2)$ is a monochromatic solution to $L(3,c)$. Otherwise $\Delta(c+2) = 1$.

If $\Delta(3c+4) = 1$, then $(c+2, c+2, 3c+4)$ is a monochromatic solution to $L(3,c)$. Otherwise $\Delta(3c+4) = 0$.

If $\Delta(2c+3) = 0$, then $(1, 2c+3, 3c+4)$ is a monochromatic solution to $L(3,c)$. Otherwise $\Delta(2c+3) = 1$.

Then if $\Delta(4c+5) = 0$, $(1, 3c+4, 4c+5)$ is a monochromatic solution to $L(3,c)$. Also, if $\Delta(4c+5) = 1$ then $(c+2, 2c+3, 4c+5)$ is a monochromatic solution to $L(3,c)$. So Δ must admit a monochromatic solution to $L(3,c)$. Then $R(3,c) \leq 4c+5$. Therefore $R(3,c) = 4c+5$. $\square$

Chapter 3

Main Result

The following two theorems are the main results of this paper.

Theorem 6. *The two-color Rado number for the equation*

$$x_1 + x_2 + x_3 + c = x_4$$

with $c < 0$ *is*

$$R(4,c) = \begin{cases} -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1 & c \text{ even} \\ \\ \infty & c \text{ odd} \end{cases}$$

Theorem 7. *The two-color Rado number for the equation*

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

with $c < 0$ *is*

$$R(5,c) = \begin{cases} -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil + 1 & c \equiv 0 \textit{ (mod 3)} \\ 7 & c = -2 \\ -\frac{c+2}{3} + 2 & -11 \le c \le -5 \textit{ and } c \equiv 1 \textit{ (mod 3)} \\ -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2 & c < -11 \textit{ and } c \equiv 1 \textit{ (mod 3)} \\ 13 & c = -1 \\ 5 & c = -4 \\ -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2 & c < -4 \textit{ and } c \equiv 2 \textit{ (mod 3)} \end{cases}$$

We prove Theorem 6 and Theorem 7 in the following chapters.

Chapter 4

m=4, c even

We restate the first case of Theorem 6 and provide a proof in this chapter.

Theorem. $R(4,c) = -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$ *for even* $c < 0$.

Proof.

4.1 Lower Bound $R(4,c) \geq -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$

Let $c < 0$ be even. We show that there exists a coloring

$\Delta'' : \left[1, -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil\right] \to [0,1]$ with no monochromatic solution to $L(4,c)$.

Let $\Delta : \left[1, 10 \cdot \left\lceil \frac{-c}{22} \right\rceil\right] \to [0,1]$ be such that Δ has no monochromatic solution to $L\left(4, 2 \cdot \left(\left\lceil \frac{-c}{22} \right\rceil - 1\right)\right)$. We know such a coloring exists since $2 \cdot \left(\left\lceil \frac{-c}{22} \right\rceil - 1\right) \geq 0$ so by Theorem 5, $R\left(4, 2 \cdot \left(\left\lceil \frac{-c}{22} \right\rceil - 1\right)\right) \geq 11 + 5 \cdot 2 \cdot \left(\left\lceil \frac{-c}{22} \right\rceil - 1\right) = 1 + 10 \cdot \left\lceil \frac{-c}{22} \right\rceil$.

Let $\Delta' : \left[1, 10 \cdot \left\lceil \frac{-c}{22} \right\rceil\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(1 + 10 \cdot \left\lceil \frac{-c}{22} \right\rceil - x\right)$. Let $\Delta'' : \left[1, -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(x + \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil}{2}\right)$ for $x \geq 1$. Suppose $\exists z_1, z_2, z_3, z_4 \in \left[1, -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil\right]$ such that

$$z_1 + z_2 + z_3 + c = z_4$$

Define y_i by $y_i = z_i + \frac{c+22\cdot\lceil\frac{-c}{22}\rceil}{2}$. Then

$$\left(y_1 - \frac{c+22\cdot\lceil\frac{-c}{22}\rceil}{2}\right) + \left(y_2 - \frac{c+22\cdot\lceil\frac{-c}{22}\rceil}{2}\right) + \left(y_3 - \frac{c+22\cdot\lceil\frac{-c}{22}\rceil}{2}\right) + c = \left(y_4 - \frac{c+22\cdot\lceil\frac{-c}{22}\rceil}{2}\right)$$

So

$$y_1 + y_2 + y_3 - \left(c + 22\cdot\left\lceil\frac{-c}{22}\right\rceil\right) + c = y_4$$
$$y_1 + y_2 + y_3 - 22\cdot\left\lceil\frac{-c}{22}\right\rceil = y_4$$

Define x_i by $x_i = 1 + 10\cdot\lceil\frac{-c}{22}\rceil - y_i$. Then

$$\left(1+10\cdot\left\lceil\frac{-c}{22}\right\rceil - x_1\right) + \left(1+10\cdot\left\lceil\frac{-c}{22}\right\rceil - x_2\right) + \left(1+10\cdot\left\lceil\frac{-c}{22}\right\rceil - x_3\right) - 22\cdot\left\lceil\frac{-c}{22}\right\rceil = \left(1+10\cdot\left\lceil\frac{-c}{22}\right\rceil - x_4\right)$$

Thus

$$2 + 20\cdot\left\lceil\frac{-c}{22}\right\rceil - x_1 - x_2 - x_3 - 22\cdot\left\lceil\frac{-c}{22}\right\rceil = -x_4$$
$$2 - 2\cdot\left\lceil\frac{-c}{22}\right\rceil - x_1 - x_2 - x_3 = -x_4$$
$$x_1 + x_2 + x_3 - 2 + 2\cdot\left\lceil\frac{-c}{22}\right\rceil = x_4$$
$$x_1 + x_2 + x_3 + 2\cdot\left(\left\lceil\frac{-c}{22}\right\rceil - 1\right) = x_4$$

Since (x_1, x_2, x_3, x_4) solves $L\left(4, 2\cdot\left(\left\lceil\frac{-c}{22}\right\rceil - 1\right)\right)$ and Δ has no monochromatic solution to $L\left(4, 2\cdot\left(\left\lceil\frac{-c}{22}\right\rceil - 1\right)\right)$, x_1, x_2, x_3, and x_4 must not be monochromatic in Δ. Consider $\Delta''(z_i) = \Delta'\left(z_i + \frac{c+22\cdot\left\lceil\frac{-c}{22}\right\rceil}{2}\right) = \Delta'(y_i) = \Delta\left(1 + 10\cdot\left\lceil\frac{-c}{22}\right\rceil - y_i\right) = \Delta(x_i)$. So z_1, z_2, z_3, and z_4 are not monochromatic in Δ''.

We have shown that there exists a coloring $\Delta'' : \left[1, -\frac{c}{2} - \left\lceil\frac{-c}{22}\right\rceil\right] \to [0,1]$ that has no monochromatic solution to $L(4,c)$. Therefore $R(4,c) \geq -\frac{c}{2} - \left\lceil\frac{-c}{22}\right\rceil + 1$ for even $c < 0$.

4.2 Upper Bound $R(4,c) \leq -\frac{c}{2} - \left\lceil\frac{-c}{22}\right\rceil + 1$

4.2.1 $c < -22$

Let c be even and $c < -22$. Let $\Delta : [1, -\frac{c}{2} - \left\lceil\frac{-c}{22}\right\rceil + 1] \to [0,1]$ be an arbitrary coloring. We show that Δ has a monochromatic solution to $L(4,c)$.

Let $\Delta' : \left[1, 10\cdot\left\lceil\frac{-c}{22}\right\rceil - 9\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+22\cdot\left\lceil\frac{-c}{22}\right\rceil - 20}{2}\right)$ for $x \geq 1$. Let $\Delta'' : \left[1, 10\cdot\left\lceil\frac{-c}{22}\right\rceil - 9\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(10\cdot\left\lceil\frac{-c}{22}\right\rceil - 8 - x\right)$. Since $R(4,c) \leq 11 + 5c$ for $c \geq 0$, $R\left(4, 2\cdot\left\lceil\frac{-c}{22}\right\rceil - 4\right) \leq 11 + 5\cdot\left(2\cdot\left\lceil\frac{-c}{22}\right\rceil - 4\right) = 10\cdot\left\lceil\frac{-c}{22}\right\rceil - 9$ since $c < -22$. So Δ'' must admit a monochromatic solution to $L\left(4, 2\cdot\left\lceil\frac{-c}{22}\right\rceil - 4\right)$. Let $z_1 + z_2 + z_3 + 2\cdot\left\lceil\frac{-c}{22}\right\rceil - 4 = z_4$ where $\Delta''(z_1) = \Delta''(z_2) = \Delta''(z_3) = \Delta''(z_4)$.

Define y_i by $y_i = 10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8 - z_i$. Then

$$\left(10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8 - y_1\right) + \left(10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8 - y_2\right) + \left(10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8 - y_3\right) \\ + 2 \cdot \left\lceil \frac{-c}{22} \right\rceil - 4 = \left(10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8 - y_4\right)$$

So

$$2 \cdot \left(10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8\right) - y_1 - y_2 - y_3 + 2 \cdot \left\lceil \frac{-c}{22} \right\rceil - 4 = -y_4$$
$$y_1 + y_2 + y_3 - 22 \cdot \left\lceil \frac{-c}{22} \right\rceil + 20 = y_4$$

Define x_i by $x_i = y_i - \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20}{2}$. Then

$$\left(x_1 + \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20}{2}\right) + \left(x_2 + \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20}{2}\right) \\ + \left(x_3 + \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20}{2}\right) - 22 \cdot \left\lceil \frac{-c}{22} \right\rceil + 20 = \left(x_4 + \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20}{2}\right)$$

Thus

$$x_1 + x_2 + x_3 + \left(c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20\right) - 22 \cdot \left\lceil \frac{-c}{22} \right\rceil + 20 = x_4$$
$$x_1 + x_2 + x_3 + c = x_4$$

Consider $\Delta(x_i) = \Delta\left(y_i - \frac{c + 22 \cdot \left\lceil \frac{-c}{22} \right\rceil - 20}{2}\right) = \Delta'(y_i) = \Delta'\left(10 \cdot \left\lceil \frac{-c}{22} \right\rceil - 8 - z_i\right) = \Delta''(z_i)$.

Since z_1, z_2, z_3 and z_4 are monochromatic in Δ'', x_1, x_2, x_3 and x_4 are monochromatic in

Δ. Therefore (x_1, x_2, x_3, x_4) is a monochromatic solution to $L(4, c)$. So $R(4, c) \leq -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$.

Therefore $R(4, c) = -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$ for c even and $c < -22$.

4.2.2 $0 > c \geq -22$

For c even and $0 > c \geq -22$, $-\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1 = -\frac{c}{2} - 1 + 1 = -\frac{c}{2}$. Let $\Delta : \left[1, -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1\right] \to [0, 1]$ and consider $x_1 = x_2 = x_3 = x_4 = -\frac{c}{2}$.

$$x_1 + x_2 + x_3 + c = -\frac{c}{2} - \frac{c}{2} - \frac{c}{2} + c = -\frac{c}{2} = x_4$$

So (x_1, x_2, x_3, x_4) is a monochromatic solution to $L(4, c)$. Then $R(4, c) \leq -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$. Therefore $R(4, c) = -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$ for c even and $0 > c \geq -22$

Then $R(4, c) = -\frac{c}{2} - \left\lceil \frac{-c}{22} \right\rceil + 1$ for all even c with $c < 0$. $\square$

Chapter 5

m=4, c odd

We restate the second case of Theorem 6 and provide a proof in this chapter.

Theorem. $R(4, c) = \infty$ *for odd* $c < 0$.

Proof.

Let $m = 4$ and $c < 0$ be odd. Consider the coloring $\Delta : \mathbb{N} \to [0, 1]$ where

$$\Delta(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

Choose x_1, x_2, x_3, and x_4 monochromatic in Δ. Then they are either all even or all odd.

If x_1, x_2, x_3, and x_4 are all even, then $x_1 + x_2 + x_3 + c$ is $\textit{even} + \textit{even} + \textit{even} + \textit{odd} = \textit{odd} \neq x_4$, so (x_1, x_2, x_3, x_4) is not a solution to $L(4, c)$.

If x_1, x_2, x_3, and x_4 are all odd, then $x_1 + x_2 + x_3 + c$ is $\textit{odd} + \textit{odd} + \textit{odd} + \textit{odd} = \textit{even} \neq x_4$, so (x_1, x_2, x_3, x_4) is not a solution to $L(4, c)$.

Then Δ does not have a monochromatic solution to $L(4, c)$. Therefore $R(4, c)$ is infinite for $m = 4$ and c odd. □

Chapter 6

$m = 5$ Special Cases

In this chapter, we prove the special cases from Theorem 7: $R(5,-1) = 13$, $R(5,-2) = 7$ and $R(5,-4) = 5$. We also prove that $R(5,-5) \leq 3$.

6.1 $c = -1$: $R(5,-1) = 13$

Proof.

6.1.1 Lower Bound

Consider $L(5,-1) : x_1 + x_2 + x_3 + x_4 - 1 = x_5$ and the coloring Δ which colors as follows:

$$\frac{0}{1}\ \frac{0}{2}\ \frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10}\ \frac{0}{11}\ \frac{0}{12}$$

We show that Δ does not admit a monochromatic solution to $L(5,-1)$.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = \Delta(x_4) = \Delta(x_5) = 1$. Then

$$x_1 + x_2 + x_3 + x_4 - 1 \geq 3 + 3 + 3 + 3 - 1 = 11 > 10 \geq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5,-1)$.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = \Delta(x_4) = \Delta(x_5) = 0$. If $x_1, x_2, x_3, x_4 \in \{1, 2\}$, then

$$x_1 + x_2 + x_3 + x_4 - 1 \geq 1 + 1 + 1 + 1 - 1 = 3$$

and

$$x_1 + x_2 + x_3 + x_4 - 1 \leq 2 + 2 + 2 + 2 - 1 = 7$$

But $x_5 \notin [3, 7]$. So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -1)$. Otherwise, $\exists x_i$ (where $1 \leq i \leq 4$) with $x_i \in \{11, 12\}$. Then

$$x_1 + x_2 + x_3 + x_4 - 1 \geq 11 + 1 + 1 + 1 - 1 = 13 > 12 \geq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -1)$.

Therefore Δ does not admit a monochromatic solution to $L(5, -1)$. We have shown that there exists a coloring $\Delta : [1, 12] \to [0, 1]$ which does not admit a monochromatic solution to $L(5, -1)$. Therefore, $R(5, -1) \geq 13$.

6.1.2 Upper Bound

Let $\Delta : [1, 13] \to [0, 1]$ be any coloring. We show that Δ admits a monochromatic solution to $L(5, -1)$. Without loss of generality, assume $\Delta(1) = 0$.

If $\Delta(3) = 0$, then $(1, 1, 1, 1, 3)$ is a monochromatic solution to $L(5, -1)$. Otherwise $\Delta(3) = 1$.

If $\Delta(11) = 1$, then $(3, 3, 3, 3, 11)$ is a monochromatic solution to $L(5, -1)$. Otherwise $\Delta(11) = 0$.

If $\Delta(5) = 0$, then $(1, 1, 5, 5, 11)$ is a monochromatic solution to $L(5, -1)$. Otherwise $\Delta(5) = 1$.

Then if $\Delta(13) = 0$, $(1, 1, 1, 11, 13)$ is a monochromatic solution to $L(5, -1)$. Also, if $\Delta(13) = 1$ then $(3, 3, 3, 5, 13)$ is a monochromatic solution to $L(5, -1)$. So Δ must admit a monochromatic solution to $L(5, -1)$. Then $R(5, -1) \leq 13$. Therefore $R(5, -1) = 13$. $\square$

6.2 $c = -2$: $R(5, -2) = 7$

Proof.

6.2.1 Lower Bound

Consider $L(5, -2) : x_1 + x_2 + x_3 + x_4 - 2 = x_5$ and the coloring Δ which colors as follows:

$$\frac{0}{1}\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{0}{6}$$

We show that Δ does not admit a monochromatic solution to $L(5, -2)$.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = \Delta(x_4) = \Delta(x_5) = 1$. Then

$$x_1 + x_2 + x_3 + x_4 - 2 \geq 2 + 2 + 2 + 2 - 2 = 6 > 5 \geq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -2)$.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = \Delta(x_4) = \Delta(x_5) = 0$. If $x_1 = x_2 = x_3 = x_4 = 1$, then

$$x_1 + x_2 + x_3 + x_4 - 2 = 1 + 1 + 1 + 1 - 2 = 2 \neq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -2)$. Otherwise, $\exists x_i$ (where $1 \leq i \leq 4$)

with $x_i = 6$. Then

$$x_1 + x_2 + x_3 + x_4 - 2 \geq 6 + 1 + 1 + 1 - 2 = 7 > 6 \geq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -2)$.

Therefore Δ does not admit a monochromatic solution to $L(5, -2)$. We have shown that there exists a coloring $\Delta : [1, 6] \to [0, 1]$ which does not admit a monochromatic solution to $L(5, -2)$. Therefore, $R(5, -2) \geq 7$.

6.2.2 Upper Bound

Let $\Delta : [1, 7] \to [0, 1]$ be any coloring. We show that Δ admits a monochromatic solution to $L(5, -2)$. Without loss of generality, assume $\Delta(1) = 0$.

If $\Delta(2) = 0$, then $(1, 1, 1, 1, 2)$ is a monochromatic solution to $L(5, -2)$. Otherwise $\Delta(2) = 1$.

If $\Delta(6) = 1$, then $(2, 2, 2, 2, 6)$ is a monochromatic solution to $L(5, -2)$. Otherwise $\Delta(6) = 0$.

If $\Delta(5) = 1$, then $(1, 1, 1, 5, 6)$ is a monochromatic solution to $L(5, -2)$. Otherwise $\Delta(5) = 0$.

If $\Delta(3) = 1$, then $(1, 1, 3, 3, 6)$ is a monochromatic solution to $L(5, -2)$. Otherwise $\Delta(3) = 0$.

Then if $\Delta(7) = 0$, $(1, 1, 1, 6, 7)$ is a monochromatic solution to $L(5, -2)$. Also, if $\Delta(7) = 1$ then $(2, 2, 2, 3, 7)$ is a monochromatic solution to $L(5, -2)$. So Δ must admit a monochromatic solution to $L(5,-2)$. Then $R(5,-2) \leq 7$. Therefore

$R(5,-2) = 7$. $\square$

6.3 $c = -4$: $R(5,-4) = 5$

Proof.

6.3.1 Lower Bound

Consider $L(5,-4) : x_1 + x_2 + x_3 + x_4 - 4 = x_5$ and the coloring Δ which colors as follows:

$$\frac{0}{1}\ \frac{1}{2}\ \frac{1}{3}\ \frac{0}{4}$$

We show that Δ does not admit a monochromatic solution to $L(5,-4)$.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = \Delta(x_4) = \Delta(x_5) = 1$. Then

$$x_1 + x_2 + x_3 + x_4 - 4 \geq 2 + 2 + 2 + 2 - 4 = 4 > 3 \geq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5,-4)$.

Let $\Delta(x_1) = \Delta(x_2) = \Delta(x_3) = \Delta(x_4) = \Delta(x_5) = 0$. If $x_1 = x_2 = x_3 = x_4 = 1$, then

$$x_1 + x_2 + x_3 + x_4 - 4 = 1 + 1 + 1 + 1 - 4 = 0 \neq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5,-4)$. Otherwise, $\exists x_i$ (where $1 \leq i \leq 4$)

with $x_i = 4$. If there is only one such x_i, then

$$x_1 + x_2 + x_3 + x_4 - 4 = 4 + 1 + 1 + 1 - 4 = 3 \neq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -4)$. Otherwise, there is more than one $x_i = 4$ (where $1 \leq i \leq 4$), then

$$x_1 + x_2 + x_3 + x_4 - 4 \geq 4 + 4 + 1 + 1 - 4 = 6 > 4 \geq x_5$$

So $(x_1, x_2, x_3, x_4, x_5)$ is not a solution to $L(5, -4)$.

Therefore Δ does not admit a monochromatic solution to $L(5, -4)$. We have shown that there exists a coloring $\Delta : [1, 4] \to [0, 1]$ which does not admit a monochromatic solution to $L(5, -4)$. Therefore, $R(5, -4) \geq 5$.

6.3.2 Upper Bound

Let $\Delta : [1, 5] \to [0, 1]$ be any coloring. We show that Δ admits a monochromatic solution to $L(5, -4)$. Without loss of generality, assume $\Delta(1) = 0$.

If $\Delta(2) = 0$, then $(1, 1, 1, 2, 1)$ is a monochromatic solution to $L(5, -4)$. Otherwise $\Delta(2) = 1$.

If $\Delta(4) = 1$, then $(2, 2, 2, 2, 4)$ is a monochromatic solution to $L(5, -4)$. Otherwise $\Delta(4) = 0$.

If $\Delta(3) = 1$, then $(1, 1, 1, 4, 3)$ is a monochromatic solution to $L(5, -4)$.

Otherwise $\Delta(3) = 0$.

Then if $\Delta(5) = 0$, $(1, 1, 1, 5, 4)$ is a monochromatic solution to $L(5, -4)$. Also, if $\Delta(5) = 1$ then $(2, 2, 2, 3, 5)$ is a monochromatic solution to $L(5, -4)$. So Δ must admit a monochromatic solution to $L(5, -4)$. Then $R(5, -4) \leq 5$. Therefore

$R(5, -4) = 5$. □

6.4 $c = -5$: $R(5, -5) \leq 3$

Proof. Let $\Delta : [1, 3] \to [0, 1]$ be any coloring. We show that Δ admits a monochromatic solution to $L(5, -5)$. Without loss of generality, assume $\Delta(1) = 0$.

If $\Delta(2) = 0$, then $(1, 1, 2, 2, 1)$ is a monochromatic solution to $L(5, -5)$.

Otherwise $\Delta(2) = 1$.

Then if $\Delta(3) = 0$, $(1, 1, 1, 3, 1)$ is a monochromatic solution to $L(5, -5)$. Also, if $\Delta(3) = 1$ then $(2, 2, 2, 2, 3)$ is a monochromatic solution to $L(5, -5)$. So Δ must admit a monochromatic solution to $L(5, -5)$. Then $R(5, -5) \leq 3$. □

6.5 Summary of Special Cases

We summarize the results from this chapter and Theorem 5 in the table below. These results will be used in the next chapter to prove the remainder of Theorem 7.

c	$R(5,c)$
≥ 0	$19 + 6c$
-1	13
-2	7
-4	5
-5	≤ 3

Table 6.1: Some Rado numbers for $L(5,c)$

Also note that $19 + 6 \cdot (-1) = 13 = R(5,-1)$ and $19 + 6 \cdot (-2) = 7 = R(5,-2)$. Then the results can also be summarized as follows:

c	$R(5,c)$
$\geq$ -2	$19 + 6c$
-4	5
-5	≤ 3

Table 6.2: Some Rado numbers for $L(5,c)$ (simplified)

Chapter 7

$m = 5$ In General

In this chapter, we prove the remaining pieces of Theorem 7 using the results summarized in Table 6.2.

7.1 $c \equiv 0 \pmod 3$

We restate the $c \equiv 0 \pmod 3$ case of Theorem 7 and provide a proof in this section.

Theorem. *$R(5,c) = -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil + 1$ for $c \equiv 0 \pmod 3$*

Proof.

7.1.1 Lower Bound: $R(5,c) \geq -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil + 1$

Let $c < 0$ and $c \equiv 0 \pmod 3$. We show that there exists a coloring $\Delta'' : \left[1, -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil\right] \to [0,1]$ with no monochromatic solution to $L(5,c)$. Let $\Delta : \left[1, 18 \cdot \left\lceil \frac{-c}{57} \right\rceil\right] \to [0,1]$ be such that Δ has no monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c}{57} \right\rceil - 3\right)$. We know such a coloring exists since $3 \cdot \left\lceil \frac{-c}{57} \right\rceil - 3 \geq 0$ so $R\left(5, 3 \cdot \left\lceil \frac{-c}{57} \right\rceil - 3\right) = 19 + 6 \cdot \left(3 \cdot \left\lceil \frac{-c}{57} \right\rceil - 3\right) = 1 + 18 \cdot \left\lceil \frac{-c}{57} \right\rceil$ by Theorem 5.

Let $\Delta' : \left[1, 18 \cdot \left\lceil \frac{-c}{57} \right\rceil\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(1 + 18 \cdot \left\lceil \frac{-c}{57} \right\rceil - x\right)$. Let $\Delta'' : \left[1, -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(x + \frac{c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil}{3}\right)$ for $x \geq 1$. Suppose $\exists z_1, z_2, z_3, z_4, z_5 \in \left[1, -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil\right]$ such that $z_1 + z_2 + z_3 + z_4 + c = z_5$.

Define y_i by $y_i = z_i + \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil}{3}$. Then

$$\left(y_1 - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil}{3}\right) + \left(y_2 - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil}{3}\right) + \left(y_3 - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil}{3}\right) + \left(y_4 - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil}{3}\right) + c = \left(y_5 - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil}{3}\right)$$

So

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 - \left(c + 57\cdot\left\lceil\frac{-c}{57}\right\rceil\right) + c &= y_5 \\ y_1 + y_2 + y_3 + y_4 - 57\cdot\left\lceil\frac{-c}{57}\right\rceil &= y_5 \end{aligned}$$

Define x_i by $x_i = 1 + 18\cdot\left\lceil\frac{-c}{57}\right\rceil - y_i$. Then

$$\left(1+18\cdot\left\lceil\frac{-c}{57}\right\rceil - x_1\right) + \left(1+18\cdot\left\lceil\frac{-c}{57}\right\rceil - x_2\right) + \left(1+18\cdot\left\lceil\frac{-c}{57}\right\rceil - x_3\right) + \left(1+18\cdot\left\lceil\frac{-c}{57}\right\rceil - x_4\right) - 57\cdot\left\lceil\frac{-c}{57}\right\rceil = \left(1+18\cdot\left\lceil\frac{-c}{57}\right\rceil - x_5\right)$$

Thus

$$\begin{aligned} 3\cdot\left(1+18\cdot\left\lceil\frac{-c}{57}\right\rceil\right) - x_1 - x_2 - x_3 - x_4 - 57\cdot\left\lceil\frac{-c}{57}\right\rceil &= -x_5 \\ x_1 + x_2 + x_3 + x_4 + 3\cdot\left\lceil\frac{-c}{57}\right\rceil - 3 &= x_5 \end{aligned}$$

Since Δ does not have a monochromatic solution to $L\left(5, 3\cdot\left\lceil\frac{-c}{57}\right\rceil - 3\right)$, x_1, x_2, x_3, x_4, and x_5 are not monochromatic in Δ. Consider

$\Delta''(z_i) = \Delta'\left(z_i + \frac{c+57\cdot\lceil\frac{-c}{57}\rceil}{3}\right) = \Delta'(y_i) = \Delta\left(1 + 18\cdot\lceil\frac{-c}{57}\rceil - y_i\right) = \Delta(x_i)$. So z_1, z_2, z_3, z_4, and z_5 are not monochromatic in Δ''. Then Δ'' has no monochromatic solution to $L(5,c)$. Therefore, $R(5,c) \geq -\frac{c}{3} - \lceil\frac{-c}{57}\rceil + 1$ for $c < 0$ and $c \equiv 0 \pmod 3$.

7.1.2 Upper Bound: $R(5,c) \leq -\frac{c}{3} - \lceil\frac{-c}{57}\rceil + 1$

$c < -57$

Let $c < -57$ and let $\Delta : \left[1, -\frac{c}{3} - \lceil\frac{-c}{57}\rceil + 1\right] \to [0,1]$. Let $\Delta' : \left[1, 18\lceil\frac{-c}{57}\rceil - 17\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+57\cdot\lceil\frac{-c}{57}\rceil - 54}{3}\right)$ for $x \geq 1$.

Let $\Delta'' : \left[1, 18\lceil\frac{-c}{57}\rceil - 17\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(18\lceil\frac{-c}{57}\rceil - 16 - x\right)$.

Also, since $c < -57$, $3\cdot\lceil\frac{-c}{57}\rceil - 6 \geq 0$. Then $R\left(5, 3\cdot\lceil\frac{-c}{57}\rceil - 6\right) = 19 + 6\cdot\left(3\cdot\lceil\frac{-c}{57}\rceil - 6\right) = 18\cdot\lceil\frac{-c}{57}\rceil - 17$. So Δ'' admits a monochromatic solution to $L\left(5, 3\cdot\lceil\frac{-c}{57}\rceil - 6\right)$. Let $z_1 + z_2 + z_3 + z_4 + 3\cdot\lceil\frac{-c}{57}\rceil - 6 = z_5$ with $\Delta''(z_1) = \Delta''(z_2) = \Delta''(z_3) = \Delta''(z_4) = \Delta''(z_5)$.

Define y_i by $y_i = 18\lceil\frac{-c}{57}\rceil - 16 - z_i$. Then

$$\left(18\cdot\left\lceil\frac{-c}{57}\right\rceil - 16 - y_1\right) + \left(18\cdot\left\lceil\frac{-c}{57}\right\rceil - 16 - y_2\right) + \left(18\cdot\left\lceil\frac{-c}{57}\right\rceil - 16 - y_3\right) + \left(18\cdot\left\lceil\frac{-c}{57}\right\rceil - 16 - y_4\right) + 3\cdot\left\lceil\frac{-c}{57}\right\rceil - 6 = \left(18\cdot\left\lceil\frac{-c}{57}\right\rceil - 16 - y_5\right)$$

So

$$3 \cdot \left(18 \cdot \left\lceil \frac{-c}{57} \right\rceil - 16\right) - y_1 - y_2 - y_3 - y_4 + 3 \cdot \left\lceil \frac{-c}{57} \right\rceil - 6 = -y_5$$

$$y_1 + y_2 + y_3 + y_4 + 54 - 57 \cdot \left\lceil \frac{-c}{57} \right\rceil = y_5$$

Define x_i by $x_i = y_i - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil-54}{3}$. Then

$$\left(x_1 + \frac{c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil - 54}{3}\right) + \left(x_2 + \frac{c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil - 54}{3}\right) + \left(x_3 + \frac{c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil - 54}{3}\right)$$
$$+ \left(x_4 + \frac{c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil - 54}{3}\right) + 54 - 57 \cdot \left\lceil \frac{-c}{57} \right\rceil = \left(x_5 + \frac{c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil - 54}{3}\right)$$

Thus

$$x_1 + x_2 + x_3 + x_4 + \left(c + 57 \cdot \left\lceil \frac{-c}{57} \right\rceil - 54\right) + 54 - 57 \cdot \left\lceil \frac{-c}{57} \right\rceil = x_5$$

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

Also, $\Delta(x_i) = \Delta\left(y_i - \frac{c+57\cdot\left\lceil\frac{-c}{57}\right\rceil-54}{3}\right) = \Delta'(y_i) = \Delta'\left(18\left\lceil\frac{-c}{57}\right\rceil - 16 - z_i\right) = \Delta''(z_i)$. Since z_1, z_2, z_3, z_4, and z_5 are monochromatic in Δ'', x_1, x_2, x_3, x_4, and x_5 are monochromatic in Δ. Therefore Δ admits a monochromatic solution to $L(5, c)$. So $R(5, c) \leq -\frac{c}{3} - \left\lceil\frac{-c}{57}\right\rceil + 1$ for $c \equiv 0 \pmod 3$ and $c < -57$.

$0 > c \geq -57$

Let $0 > c \geq -57$ and $\Delta : \left[1, -\frac{c}{3}\right] \to [0,1]$. Consider $x_1 = x_2 = x_3 = x_4 = x_5 = -\frac{c}{3}$. Then

$$x_1 + x_2 + x_3 + x_4 + c = -\frac{c}{3} - \frac{c}{3} - \frac{c}{3} - \frac{c}{3} + c = -\frac{c}{3} = x_5$$

So $\left(-\frac{c}{3}, -\frac{c}{3}, -\frac{c}{3}, -\frac{c}{3}, -\frac{c}{3}\right)$ is a monochromatic solution to $L(5,c)$. Then $R(5,c) \leq -\frac{c}{3} = -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil + 1$ for $0 > c \geq -57$ and $c \equiv 0 \pmod 3$.

Therefore, $R(5,c) = -\frac{c}{3} - \left\lceil \frac{-c}{57} \right\rceil + 1$ for all $c < 0$ with $c \equiv 0 \pmod 3$. □

7.2 $c \equiv 1$ (mod 3)

We restate the remaining $c \equiv 1 \pmod 3$ cases of Theorem 7 and provide a proof in this section.

Theorem.

$$R(5,c) = \begin{cases} -\frac{c+2}{3} + 2 & -11 \leq c \leq -5 \textit{ and } c \equiv 1 \textit{ (mod 3)} \\ -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2 & c < -11 \textit{ and } c \equiv 1 \textit{ (mod 3)} \end{cases}$$

We first prove the case where $-11 \leq c \leq -5$ and $c \equiv 1 \pmod 3$.

7.2.1 $-5 \geq c \geq -11$: $R(5,c) = -\frac{c+2}{3} + 2$

Proof.

Lower Bound: $R(5, c) \geq 2 - \frac{c+2}{3}$

Let $-5 \geq c \geq -11$ where $c \equiv 1 \pmod 3$ and let $\Delta : [1, 4] \to [0, 1]$ be such that Δ has no monochromatic solution to $L(5, -4)$. Let $\Delta' : [1, 4] \to [0, 1]$ be defined by $\Delta'(x) = \Delta(5 - x)$.

Let $\Delta'' : \left[1, 1 - \frac{c+2}{3}\right] \to [0, 1]$ be defined by $\Delta''(x) = \Delta'\left(x + \frac{c+11}{3}\right)$ for $x \geq 1$. Suppose $\exists z_1, z_2, z_3, z_4, z_5 \in \left[1, 1 - \frac{c+2}{3}\right]$ such that $z_1 + z_2 + z_3 + z_4 + c = z_5$.

Define y_i by $y_i = z_i + \frac{c+11}{3}$. Then

$$\left(y_1 - \frac{c+11}{3}\right) + \left(y_2 - \frac{c+11}{3}\right) + \left(y_3 - \frac{c+11}{3}\right) + \left(y_4 - \frac{c+11}{3}\right) + c = \left(y_5 - \frac{c+11}{3}\right)$$

So

$$y_1 + y_2 + y_3 + y_4 - (c + 11) + c = y_5$$

$$y_1 + y_2 + y_3 + y_4 - 11 = y_5$$

Define x_i by $x_i = 5 - y_i$. Then

$$(5 - x_1) + (5 - x_2) + (5 - x_3) + (5 - x_4) - 11 = (5 - x_5)$$

$$15 - x_1 - x_2 - x_3 - x_4 - 11 = -x_5$$

$$x_1 + x_2 + x_3 + x_4 - 4 = x_5$$

Since Δ does not admit a monochromatic solution to $L(5,-4)$, x_1, x_2, x_3, x_4, and x_5 are not monochromatic in Δ. Also, $\Delta(x_i) = \Delta(5 - y_i) = \Delta'(y_i) = \Delta'\left(z_i + \frac{c+11}{3}\right) = \Delta''(z_i)$ so z_1, z_2, z_3, z_4, and z_5 are not monochromatic in Δ''. Then Δ'' does not admit a monochromatic solution to $L(5,c)$. Therefore $R(5,c) \geq 2 - \frac{c+2}{3}$ for $c \equiv 1 \pmod 3$ and $-5 \geq c \geq -11$.

Upper Bound: $R(5,c) \leq 2 - \frac{c+2}{3}$

Let $-5 \geq c \geq -11$ where $c \equiv 1 \pmod 3$ and let $\Delta : \left[1, 2 - \frac{c+2}{3}\right] \to [0,1]$. Let $\Delta' : [1,3] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+5}{3}\right)$ for $x \geq 1$. Since $R(5,-5) \leq 3$ by results in Table 6.2, Δ' admits a monochromatic solution to $L(5,-5)$. Let $y_1 + y_2 + y_3 + y_4 - 5 = y_5$ with $\Delta'(y_1) = \Delta'(y_2) = \Delta'(y_3) = \Delta'(y_4) = \Delta'(y_5)$.

Define x_i by $x_i = y_i - \frac{c+5}{3}$. Then

$$\left(x_1 + \frac{c+5}{3}\right) + \left(x_2 + \frac{c+5}{3}\right) + \left(x_3 + \frac{c+5}{3}\right) + \left(x_4 + \frac{c+5}{3}\right) - 5 = \left(x_5 + \frac{c+5}{3}\right)$$

$$x_1 + x_2 + x_3 + x_4 + (c+5) - 5 = x_5$$

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

Also, $\Delta(x_i) = \Delta\left(y_i - \frac{c+5}{3}\right) = \Delta'(y_i)$. Since y_1, y_2, y_3, y_4, and y_5 are monochromatic in Δ', x_1, x_2, x_3, x_4, and x_5 are monochromatic in Δ. Then Δ admits a monochromatic solution to $L(5,c)$. So $R(5,c) \leq 2 - \frac{c+2}{3}$ for $-5 \geq c \geq -11$ and $c \equiv 1 \pmod 3$.

Therefore, $R(5,c) = 2 - \frac{c+2}{3}$ for $-5 \geq c \geq -11$ and $c \equiv 1 \pmod 3$. $\square$

7.2.2 $c < -11$: $R(5,c) = -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2$

We now prove the case where $c < -11$.

Proof.

Lower Bound: $R(5,c) \geq -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2$

Let $c < -11$ and $c \equiv 1 \pmod 3$. We show that there exists a coloring $\Delta'' : \left[1, -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 1\right] \to [0,1]$ with no monochromatic solution to $L\,(5,c)$. Let $\Delta : \left[1, 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 6\right] \to [0,1]$ be such that Δ has no monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 4\right)$. We know such a coloring exists since $3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 4 \geq -1$ for $c < -11$ so $R\left(5, 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 4\right) = 19 + 6 \cdot \left(3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 4\right) = 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5$ by the results in Table 6.2. Let $\Delta' : \left[1, 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 6\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - x\right)$. Let $\Delta'' : \left[1, -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 1\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(x + \frac{c+57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}\right)$ for $x \geq 1$. Suppose $\exists z_1, z_2, z_3, z_4, z_5 \in \left[1, -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 1\right]$ such that $z_1 + z_2 + z_3 + z_4 + c = z_5$.

Define y_i by $y_i = z_i + \frac{c+57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}$. Then

$$\begin{aligned}
\left(y_1 - \frac{c + 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}\right) + \left(y_2 - \frac{c + 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}\right) \\
+ \left(y_3 - \frac{c + 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}\right) + \left(y_4 - \frac{c + 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}\right) + c \\
= \left(y_5 - \frac{c + 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19}{3}\right)
\end{aligned}$$

So

$$y_1 + y_2 + y_3 + y_4 - \left(c + 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 19\right) + c = y_5$$
$$y_1 + y_2 + y_3 + y_4 - 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil + 19 = y_5$$

Define x_i by $x_i = 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - y_i$. Then

$$\left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - x_1\right) + \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - x_2\right)$$
$$+ \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - x_3\right) + \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - x_4\right)$$
$$- 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil + 19 = \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - x_5\right)$$

Thus

$$3 \cdot \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5\right) - x_1 - x_2 - x_3 - x_4 - 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil + 19 = -x_5$$
$$x_1 + x_2 + x_3 + x_4 + 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 4 = x_5$$

Since Δ does not have a monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 4\right)$, x_1, x_2, x_3, x_4, and x_5 are not monochromatic in Δ. Consider $\Delta''(z_i) = \Delta'\left(z_i + \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-19}{3}\right) = \Delta'(y_i) = \Delta\left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 5 - y_i\right) = \Delta(x_i)$. So z_1, z_2, z_3, z_4, and z_5 are not monochromatic in Δ''. Then Δ'' has no monochromatic solution to $L(5, c)$. Therefore, $R(5, c) \geq -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2$ for $c < -11$ and $c \equiv 1 \pmod 3$.

Upper Bound: $R(5,c) \leq -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2$

$c < -38$

Let $c < -38$ and $c \equiv 1 \pmod 3$. Let $\Delta : \left[1, -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2\right] \to [0,1]$. Let $\Delta' : \left[1, 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 23\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+57\cdot\left\lceil \frac{-c+19}{57} \right\rceil - 73}{3}\right)$ for $x \geq 1$. Let $\Delta'' : \left[1, 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 23\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - x\right)$. Note that for $c < -38$, $3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7 \geq -1$. So $R\left(5, 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7\right) = 19 + 6 \cdot \left(3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7\right) = 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 23$ by results in Table 6.2. Then Δ'' admits a monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7\right)$. Let $z_1 + z_2 + z_3 + z_4 + 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7 = z_5$ with $\Delta''(z_1) = \Delta''(z_2) = \Delta''(z_3) = \Delta''(z_4) = \Delta''(z_5)$.

Define y_i by $y_i = 18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - z_i$. Then

$$\begin{aligned}\left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - y_1\right) + \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - y_2\right) \\ + \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - y_3\right) + \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - y_4\right) \\ + 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7 = \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22 - y_5\right)\end{aligned}$$

So

$$\begin{aligned}3 \cdot \left(18 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 22\right) - y_1 - y_2 - y_3 - y_4 + 3 \cdot \left\lceil \frac{-c+19}{57} \right\rceil - 7 = -y_5 \\ y_1 + y_2 + y_3 + y_4 - 57 \cdot \left\lceil \frac{-c+19}{57} \right\rceil + 73 = y_5\end{aligned}$$

Define x_i by $x_i = y_i - \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}$. Then

$$\left(x_1 + \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}\right) + \left(x_2 + \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}\right) + \left(x_3 + \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}\right) + \left(x_4 + \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}\right) - 57\cdot\left\lceil\frac{-c+19}{57}\right\rceil + 73 = \left(x_5 + \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}\right)$$

Thus

$$x_1 + x_2 + x_3 + x_4 + \left(c + 57\cdot\left\lceil\frac{-c+19}{57}\right\rceil - 73\right) - 57\cdot\left\lceil\frac{-c+19}{57}\right\rceil + 73 = x_5$$

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

Also,

$\Delta(x_i) = \Delta\left(y_i - \frac{c+57\cdot\left\lceil\frac{-c+19}{57}\right\rceil-73}{3}\right) = \Delta'(y_i) = \Delta'\left(18\cdot\left\lceil\frac{-c+19}{57}\right\rceil - 22 - z_i\right) = \Delta''(z_i)$.

Since z_1, z_2, z_3, z_4, and z_5 are monochromatic in Δ'', x_1, x_2, x_3, x_4, and x_5 are monochromatic in Δ. So Δ admits a monochromatic solution to $L(5, c)$. Then $R(5, c) \leq -\frac{c+2}{3} - \left\lceil\frac{-c+19}{57}\right\rceil + 2$ for $c < -38$ and $c \equiv 1 \pmod 3$.

Therefore, $R(5, c) = -\frac{c+2}{3} - \left\lceil\frac{-c+19}{57}\right\rceil + 2$ for $c < -38$ and $c \equiv 1 \pmod 3$.

$-11 > c \geq -38$

Let $-11 > c \geq -38$ and $c \equiv 1 \pmod 3$. Let $\Delta : \left[1, -\frac{c+2}{3} + 1\right] \to [0, 1]$ and let $\Delta' : [1, 5] \to [0, 1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+14}{3}\right)$ for $x \geq 1$. Let $\Delta'' : [1, 5] \to [0, 1]$ be defined by $\Delta''(x) = \Delta'(6 - x)$. Since $R(5, -4) = 5$ by results in Table 6.2, Δ''

admits a monochromatic solution to $L(5,-4)$. Let $z_1+z_2+z_3+z_4-4=z_5$ with $\Delta''(z_1)=\Delta''(z_2)=\Delta''(z_3)=\Delta''(z_4)=\Delta''(z_5)$.

Define y_i by $y_i=6-z_i$. Then

$$(6-y_1)+(6-y_2)+(6-y_3)+(6-y_4)-4=(6-y_5)$$
$$18-y_1-y_2-y_3-y_4-4=-y_5$$
$$y_1+y_2+y_3+y_4-14=-y_5$$

Define x_i by $x_i=y_i-\frac{c+14}{3}$. Then

$$\left(x_1+\frac{c+14}{3}\right)+\left(x_2+\frac{c+14}{3}\right)+\left(x_3+\frac{c+14}{3}\right)+\left(x_4+\frac{c+14}{3}\right)-14 = \left(x_5+\frac{c+14}{3}\right)$$

So

$$x_1+x_2+x_3+x_4+(c+14)-14=x_5$$
$$x_1+x_2+x_3+x_4+c=x_5$$

Also, $\Delta(x_i)=\Delta\left(y_i-\frac{c+14}{3}\right)=\Delta'(y_i)=\Delta'(6-z_i)=\Delta''(z_i)$. Since z_1,z_2,z_3,z_4, and z_5 are monochromatic in Δ'', x_1,x_2,x_3,x_4, and x_5 are monochromatic in Δ. So Δ admits a monochromatic solution to $L(5,c)$. Then $R(5,c)\leq -\frac{c+2}{3}+1$ for $-11>c\geq -38$ and $c\equiv 1 \pmod 3$.

Note that $-\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2 = -\frac{c+2}{3} + 1$ for $-11 > c \geq -38$ and $c \equiv 1 \pmod 3$. Therefore, $R(5,c) = -\frac{c+2}{3} - \left\lceil \frac{-c+19}{57} \right\rceil + 2$ for all $c < -11$ and $c \equiv 1 \pmod 3$. □

7.3 $c \equiv 2$ (mod 3)

We restate the remaining $c \equiv 2 \pmod 3$ case of Theorem 7 and provide a proof in this section.

Theorem. $R(5,c) = -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2$ *for* $c < -4$ *and* $c \equiv 2$ *(mod 3)*

Proof.

7.3.1 Lower Bound: $R(5,c) \geq -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2$

Let $c < -4$ and $c \equiv 2 \pmod 3$. We show that there exists a coloring $\Delta'' : \left[1, -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 1\right] \to [0,1]$ with no monochromatic solution to $L(5,c)$. Let $\Delta : \left[1, 18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 12\right] \to [0,1]$ be such that Δ has no monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 5\right)$. We know such a coloring exists since $3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 5 \geq -2$ for $c < -4$ so $R\left(5, 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 5\right) \geq 19 + 6 \cdot \left(3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 5\right) = 18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 11$ by results in Table 6.2. Let $\Delta' : \left[1, 18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 12\right] \to [0,1]$ be defined by $\Delta'(x) = \Delta\left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 11 - x\right)$. Let $\Delta'' : \left[1, -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 1\right] \to [0,1]$ be defined by $\Delta''(x) = \Delta'\left(x + \frac{c + 57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 38}{3}\right)$ for $x \geq 1$. Suppose $\exists z_1, z_2, z_3, z_4, z_5 \in \left[1, -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 1\right]$ such that $z_1 + z_2 + z_3 + z_4 + c = z_5$.

Define y_i by $y_i = z_i + \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil-38}{3}$. Then

$$\left(y_1 - \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil-38}{3}\right) + \left(y_2 - \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil-38}{3}\right) + \left(y_3 - \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil-38}{3}\right) + \left(y_4 - \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil-38}{3}\right) + c = \left(y_5 - \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil-38}{3}\right)$$

So

$$y_1 + y_2 + y_3 + y_4 - \left(c + 57\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 38\right) + c = y_5$$
$$y_1 + y_2 + y_3 + y_4 - 57\cdot\left\lceil\frac{-c+38}{57}\right\rceil + 38 = y_5$$

Define x_i by $x_i = 18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11 - y_i$. Then

$$\left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11 - x_1\right) + \left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11 - x_2\right) + \left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11 - x_3\right) + \left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11 - x_4\right) - 57\cdot\left\lceil\frac{-c+38}{57}\right\rceil + 38 = \left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11 - x_5\right)$$

Thus

$$3\cdot\left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 11\right) - x_1 - x_2 - x_3 - x_4 - 57\cdot\left\lceil\frac{-c+38}{57}\right\rceil + 38 = -x_5$$
$$x_1 + x_2 + x_3 + x_4 + 3\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 5 = x_5$$

Since Δ does not have a monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 5\right)$, x_1, x_2, x_3, x_4, and x_5 are not monochromatic in Δ. Consider $\Delta''(z_i) = \Delta'\left(z_i + \frac{c+57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 38}{3}\right) = \Delta'(y_i) = \Delta\left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 11 - y_i\right) = \Delta(x_i)$. So z_1, z_2, z_3, z_4, and z_5 are not monochromatic in Δ''. Then Δ'' has no monochromatic solution to $L(5, c)$. Therefore, $R(5, c) \geq -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2$ for $c < -4$ and $c \equiv 2 \pmod 3$.

7.3.2 Upper Bound: $R(5, c) \leq -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2$

$c < -19$

Let $c < -19$ and $c \equiv 2 \pmod 3$. Let $\Delta : \left[1, -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2\right] \to [0, 1]$. Let $\Delta' : \left[1, 18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 29\right] \to [0, 1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 92}{3}\right)$ for $x \geq 1$. Let $\Delta'' : \left[1, 18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 29\right] \to [0, 1]$ be defined by $\Delta''(x) = \Delta'\left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28 - x\right)$. Note that for $c < -19$, $3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8 \geq -2$. So $R\left(5, 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8\right) = 19 + 6 \cdot \left(3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8\right) = 18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 29$ by results in Table 6.2. So Δ'' admits a monochromatic solution to $L\left(5, 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8\right)$. Let $z_1 + z_2 + z_3 + z_4 + 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8 = z_5$ with $\Delta''(z_1) = \Delta''(z_2) = \Delta''(z_3) = \Delta''(z_4) = \Delta''(z_5)$.

Define y_i by $y_i = 18 \cdot \lceil \frac{-c+38}{57} \rceil - 28 - z_i$. Then

$$\left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28 - y_1\right) + \left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28 - y_2\right) + \left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28 - y_3\right) + \left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28 - y_4\right) + 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8 = \left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28 - y_5\right)$$

So

$$3 \cdot \left(18 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 28\right) - y_1 - y_2 - y_3 - y_4 + 3 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 8 = -y_5$$

$$y_1 + y_2 + y_3 + y_4 - 57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil + 92 = y_5$$

Define x_i by $x_i = y_i - \frac{c+57 \cdot \lceil \frac{-c+38}{57} \rceil - 92}{3}$. Then

$$\left(x_1 + \frac{c+57 \cdot \lceil \frac{-c+38}{57} \rceil - 92}{3}\right) + \left(x_2 + \frac{c+57 \cdot \lceil \frac{-c+38}{57} \rceil - 92}{3}\right) + \left(x_3 + \frac{c+57 \cdot \lceil \frac{-c+38}{57} \rceil - 92}{3}\right) + \left(x_4 + \frac{c+57 \cdot \lceil \frac{-c+38}{57} \rceil - 92}{3}\right) - 57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil + 92 = \left(x_5 + \frac{c+57 \cdot \lceil \frac{-c+38}{57} \rceil - 92}{3}\right)$$

Thus

$$x_1 + x_2 + x_3 + x_4 + \left(c + 57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil - 92\right) - 57 \cdot \left\lceil \frac{-c+38}{57} \right\rceil + 92 = x_5$$

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

Also,

$\Delta(x_i) = \Delta\left(y_i - \frac{c+57\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 92}{3}\right) = \Delta'(y_i) = \Delta'\left(18\cdot\left\lceil\frac{-c+38}{57}\right\rceil - 28 - z_i\right) = \Delta''(z_i)$.

Since z_1, z_2, z_3, z_4, and z_5 are monochromatic in Δ'', x_1, x_2, x_3, x_4, and x_5 are monochromatic in Δ. So Δ admits a monochromatic solution to $L(5, c)$. Then $R(5, c) \leq -\frac{c+1}{3} - \left\lceil\frac{-c+38}{57}\right\rceil + 2$ for $c < -19$ and $c \equiv 2 \pmod 3$.

Therefore, $R(5, c) = -\frac{c+1}{3} - \left\lceil\frac{-c+38}{57}\right\rceil + 2$ for $c < -19$ and $c \equiv 2 \pmod 3$.

$-4 > c \geq -19$

Let $-4 > c \geq -19$ and $c \equiv 2 \pmod 3$. Let $\Delta : \left[1, -\frac{c+1}{3} + 1\right] \to [0, 1]$ and let $\Delta' : [1, 3] \to [0, 1]$ be defined by $\Delta'(x) = \Delta\left(x - \frac{c+7}{3}\right)$ for $x \geq 1$. Let $\Delta'' : [1, 3] \to [0, 1]$ be defined by $\Delta''(x) = \Delta'(4 - x)$. Since $R(5, -5) \leq 3$ by results in Table 6.2, Δ'' admits a monochromatic solution to $L(5, -5)$. Let $z_1 + z_2 + z_3 + z_4 - 5 = z_5$ with $\Delta''(z_1) = \Delta''(z_2) = \Delta''(z_3) = \Delta''(z_4) = \Delta''(z_5)$.

Define y_i by $y_i = 4 - z_i$. Then

$$(4 - y_1) + (4 - y_2) + (4 - y_3) + (4 - y_4) - 5 = (4 - y_5)$$

$$12 - y_1 - y_2 - y_3 - y_4 - 5 = -y_5$$

$$y_1 + y_2 + y_3 + y_4 - 7 = -y_5$$

Define x_i by $x_i = y_i - \frac{c+7}{3}$. Then

$$\left(x_1 + \frac{c+7}{3}\right) + \left(x_2 + \frac{c+7}{3}\right) + \left(x_3 + \frac{c+7}{3}\right) + \left(x_4 + \frac{c+7}{3}\right) - 7 = \left(x_5 + \frac{c+7}{3}\right)$$

So

$$x_1 + x_2 + x_3 + x_4 + (c+7) - 7 = x_5$$

$$x_1 + x_2 + x_3 + x_4 + c = x_5$$

Also, $\Delta(x_i) = \Delta\left(y_i - \frac{c+7}{3}\right) = \Delta'(y_i) = \Delta'(4 - z_i) = \Delta''(z_i)$. Since z_1, z_2, z_3, z_4, and z_5 are monochromatic in Δ'', x_1, x_2, x_3, x_4, and x_5 are monochromatic in Δ. So Δ admits a monochromatic solution to $L(5, c)$. Then $R(5, c) \leq -\frac{c+1}{3} + 1$ for $-4 > c \geq -19$ and $c \equiv 2 \pmod 3$.

Note that $-\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2 = -\frac{c+2}{3} + 1$ for $-4 > c \geq -19$ and $c \equiv 2 \pmod 3$. Therefore, $R(5, c) = -\frac{c+1}{3} - \left\lceil \frac{-c+38}{57} \right\rceil + 2$ for all $c < -4$ and $c \equiv 2 \pmod 3$. $\square$

Chapter 8

Suggestions for Further Research

This problem may be investigated further by considering $m = 6, 7, \ldots$ for $c < 0$. Although a generalization may be possible, the formula seems to become more complicated with each larger value of m. We also suspect that the quantity of special cases will increase with larger values of m.

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