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Power Radiation and Dissipation in the Conducting Medium Around a Small loop Antenna

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POWER RADIATION AND DISSIPATION IN THE CONDUCTING MEDIUM

AROUND A SMALL LOOP ANTENNA

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This thesis is approved as a creditable, independent investigation by a candidate for the degree, Master of Science, and acceptable as meeting the thesis requirements for this degree; but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Advisor

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CHAPTER I

INTRODUCTION

There is considerable interest in the study of electromagnetic systems which are immersed in a conducting medium. This interest originates from practical applications such as communications among submarines under the sea surface, or measurement of field intensity or frequency change of the wave transmitted from a transmitter placed inside the body of an animal. The radiation from antennas immersed in a conducting medium will be radically different from those immersed in a nondissipative medium. The radiated fields from the antenna suffers additional attenuation and phase distortion as a result of the finite conductivity of the medium. The power relationships are greatly modified from the case of free space.

According to Stratton[1], the presence of conductivity introduces serious analytical difficulties. However, with the aid of the already known field functions[2] the general expressions of the total energy radiated and the total energy dissipated in the conducting medium can be determined. The aim of this thesis is to investigate the nature of the electromagnetic wave transmitted from a loop antenna enclosed by a spherical insulating cavity which is immersed in a conducting medium.

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[1]Julius Adams Stratton, Electromagnetic Theory, p. 424, McGraw-Hill Book Co., Inc., New York, 1941.

[2]James R. Wait, The Magnetic Dipole Antenna Immersed in a Conducting Medium, Proc., I. R. E., October, 1952.

Two general formulas for power radiation and dissipation have been derived under the following assumptions:

1. The conducting medium outside the spherical insulating cavity was everywhere homogeneous with infinite dimension.

2. The peripherical length or the size of the loop was very small compared with the wave length of the radiated fields.

3. The conventional spherical polar system of coordinates (r, θ, ϕ) was chosen with the loop situated at the origin and oriented in the polar direction as illustrated in Figure 1.

Rationalized M. K. S. system of units have been used.

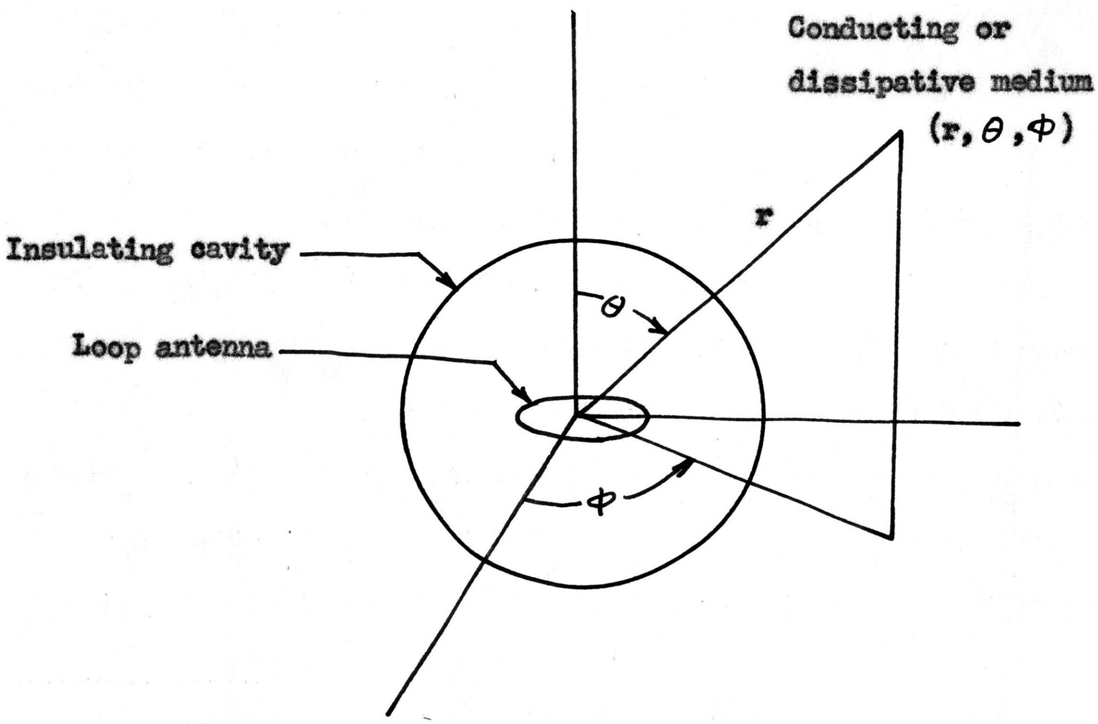

Figure 1. Spherical Coordinates System with a Small Loop Antenna at the Origin

TABLE I. SYMBOLS USED

E	Electric field intensity vector, volts per meter.
H	Magnetic field intensity vector, amperes per meter.
i	Current density vector, amperes per square meter.
j	$\sqrt{-1}$
P_r	Total outward flow of power, watts.
P_d	Total dissipated power, watts.
P	Power dissipated per unit volume, watts per cubic meter.
S_r	Average value of Poynting's vector, watts per square meter.
t	Time in seconds.
∇^2	Laplacian operator.
σ	Conductivity of the medium, mhos per meter.
ϵ	Permittivity of the medium, farads per meter.
μ	Permeability of the medium, henries per meter.
θ	Polar angle of spherical coordinates, radians.
ϕ	Azimuth angle of spherical coordinates, radians.
ψ	Scalar or vector wave function.
w	Angular velocity, radians per second.
$\times$	Vector cross product.
$\bullet$	Vector dot product.
$\triangleq$	Equal by definition.

CHAPTER II

DERIVATION OF FIELD EQUATIONS

In this chapter, under the assumptions listed in the first chapter, following the general procedure used by Wait[3], the fundamental electromagnetic field equations for a conducting medium are derived. The general expressions for electric and magnetic fields in spherical polar coordinates are derived from Maxwell's equations.

The Wave Equation

The following expression is a general form of the wave equation[4].

$$\nabla^2 \psi = K_1^2 \frac{\partial^2 \psi}{\partial t^2} + K_2 \frac{\partial \psi}{\partial t} + g(x,y,z,t) \qquad (2\text{-}1)$$

In this expression, wave function (x,y,z,t) can be either a scalar or a vector function, which we assume to be differentiable, and K_1 and K_2 are real positive constants. The function $g(x,y,z,t)$ usually represents a source.

If the source of disturbance or forcing function does not exist inside the region under consideration, the general form (2-1) becomes

$$\nabla^2 \psi = K_1^2 \frac{\partial^2 \psi}{\partial t^2} + K_2 \frac{\partial \psi}{\partial t} \qquad (2\text{-}2)$$

Where the term $K_2 \frac{\partial \psi}{\partial \psi}$ represents the effect of damping in a passive system. This is the type of wave equation treated in this paper.

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[3] Ibid.

[4] Arthur Bronwell, Advanced Mathematics in Physics and Engineering, Chapter 2, McGraw-Hill Book Co., Inc.: New York, 1953.

The Wave Equation Derived from Maxwell's Equations

Maxwell's equations in a uniform material having permeability μ, permittivity ϵ, conductivity σ, but not any charge, or any current other than that determined by Ohm's law are

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \sigma E + \epsilon \frac{\partial E}{\partial t}$$

$$\nabla \cdot H = 0$$

$$\nabla \cdot E = 0$$

Taking the curl of the first equation and substituting $\nabla \times H$ from the second equation one obtains

$$\nabla \times \nabla \times E = -\sigma\mu \frac{\partial E}{\partial t} - \epsilon\mu \frac{\partial^2 E}{\partial t^2}$$

Similarly

$$\nabla \times \nabla \times H = -\sigma\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\partial^2 H}{\partial t^2}$$

Using the equation of vector analysis,

$$\nabla \times \nabla \times A = \nabla\nabla \cdot A - \nabla^2 A$$

where A is an arbitrary vector function, and using the equations

$E = 0$, $H = 0$, these equations become

$$\nabla^2 E - \sigma\mu \frac{\partial E}{\partial t} - \epsilon\mu \frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 H - \sigma\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\partial^2 H}{\partial t^2} = 0 \qquad (2\text{-}3)$$

Thus E and H satisfy the same wave equation of the form as shown in equation (2-2). The equations (2-3) are vector equations, which means each of the three components of E and the three components of H separately satisfies the same scalar wave equation. Then a wave equation of the form

$$\nabla^2 \psi - \sigma\mu \frac{\partial \psi}{\partial t} - \epsilon\mu \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad (2\text{-}4)$$

Where ψ, a scalar, can stand for one of the components of E or H. Since an arbitrary time variation of the field can be represented by Fourier analysis in terms of harmonic components, no essential loss of generality will be incurred by the assumption that

$$\psi \triangleq (r, \theta, \psi, t) = f(r, \theta, \phi)\, e^{jwt} \qquad (2\text{-}5)$$

in spherical coordinates. Where $j = \sqrt{-1}$ and w, the angular velocity, is a constant. Substituting (2-5) in the wave equation (2-4), we get

$$\nabla^2 \psi - (jw\sigma\mu - w^2\epsilon\mu)\psi = 0$$

or $$\nabla^2 \psi - (jw\mu)(\sigma - jw\epsilon) = 0$$

The above equation may be written in the form

$$\nabla^2 E - \gamma^2 E = 0 \qquad (2\text{-}6)$$

where $$\gamma^2 = jw\mu(\sigma + jw\epsilon) \qquad (2\text{-}7)$$

The constant γ is known as the propagation constant for the wave. In general, γ is complex and has real and imaginary parts designated by α and β respectively. That is $\gamma = \alpha + j\beta$. It can be shown[5] that

$$\alpha = \operatorname{Re} \sqrt{j(w\mu)(\sigma + jw\epsilon)}$$

$$= \omega \sqrt{\frac{\mu\epsilon}{2}} \sqrt{\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1} \qquad (2\text{-}8)$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2}} \sqrt{\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1} \qquad (2\text{-}9)$$

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[5]See Appendix A.

Circular Electric Waves[6] Outside the Insulating Cavity

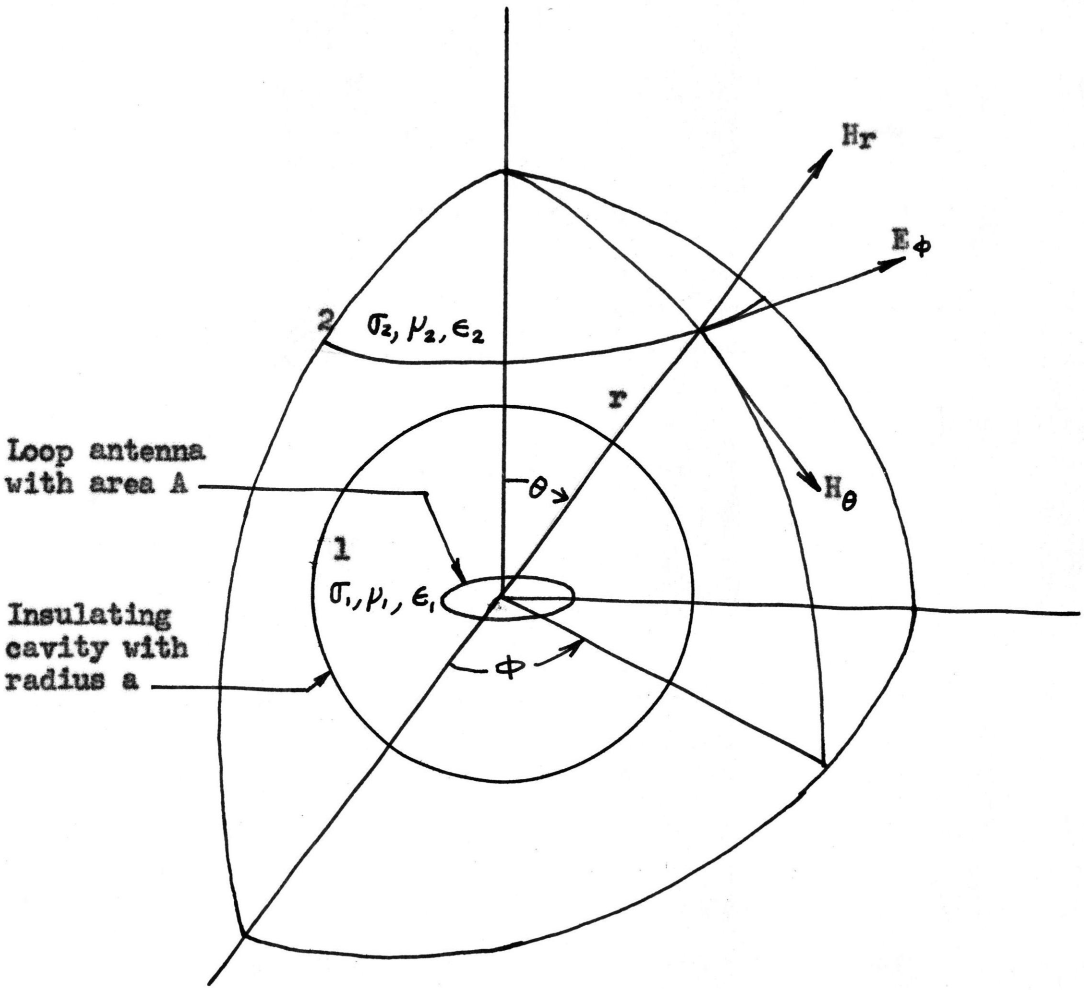

Figure 2. Field Vectors Produced by a Small Loop Antenna Situated at the Origin of a Spherical Coordinates System

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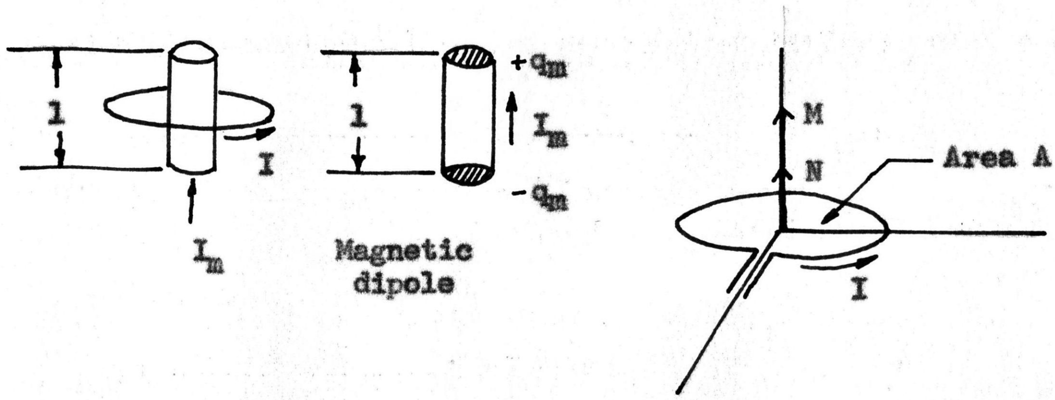
[6] S. A. Schelkunoff, Electromagnetic Waves, Sec. 8.15, D. Van Nostrand Co., Inc., New York, 1943.



$I_m = -\mu \frac{dq_m}{dt}$ = Fictitious magnetic current = $I_{mo}\, e^{jwt}$

I = $I_o\, e^{jwt}$ = Uniform inphase current in the loop

q_m = Magnetic charge

M = Magnetic dipole moment $\triangleq q_m l = IAN$

N = Unit vector of the area A

Figure 3. Equivalent Magnetic Dipole

As shown in Figure 3, a small loop carrying current I may be conceived of as an oscillating magnetic dipole with magnetic moment equal to IA.[7] The wave equation of an oscillating magnetic dipole has been solved under the condition of uniform current distribution, that is, when the size of the magnetic dipole or the loop is very small as compared with the wave length.

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[7]John D. Kraus, Antennas, chapter 6, McGraw-Hill Book Co., Inc., 1950.

The wave equation (2-6) expressed in spherical polar coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}-\gamma^2\psi=0 \tag{2-10}$$

It is evident that the circular loop, as shown in Figure 2, generates circular transverse electric waves independent of ϕ,[8] therefore, $\frac{\partial\psi}{\partial\phi}=0$ and (2-10) can be simplified as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\gamma^2\psi=0 \tag{2-10a}$$

The resulting fields are given[9] by the following equations in terms of a scalar wave function ψ_m:

$$H_{rm}=\frac{1}{j\mu_m w}\left(\frac{\partial^2\psi_m}{\partial r^2}-\gamma_m^2\psi_m\right) \tag{2-11}$$

$$E_{\phi m}=\frac{1}{r}\frac{\partial^2\psi_m}{\partial r\,\partial\theta} \tag{2-12}$$

$$H_{\theta m}=\frac{1}{j\mu_m wr}\frac{\partial^2\psi_m}{\partial r\,\partial\theta} \tag{2-13}$$

$$E_{rm}=E_{\theta m}=H_{\phi m}=0$$

where

$$\gamma_m^2=j\sigma_m\mu_m w-\epsilon_m\mu_m w^2$$

and the subscript m takes the value 1 or 2 to denote the region interior or exterior of the insulating cavity, respectively. The function ψ_m

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[8] Stratton, op. cit., Chapter 8.

[9] Donald H. Menzel, Fundamental Formulas of Physics, Vol. 1, Sec. 5-6, Dover Publications Inc., New York, 1960.

satisfies the equation[10]

$$r^2 \frac{\partial^2 \psi_m}{\partial r^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial \psi_m}{\partial\theta}) - \gamma_m r^2 \psi_m = 0 \qquad (2\text{-}14)$$

Now the primary field inside the cavity is well known[11] and the field components are given by

$$H^o_{r1} = \frac{IA}{2\pi} (\frac{1}{r^3} + \frac{\gamma_1}{r^2}) e^{-\gamma_1 r} \cos\theta \qquad (2\text{-}15)$$

$$H^o_{\theta 1} = \frac{IA}{4\pi} (\frac{1}{r^3} + \frac{\gamma_1}{r^2} + \frac{\gamma_1^2}{r}) e^{-\gamma_1 r} \sin\theta \qquad (2\text{-}16)$$

$$E^o_{\phi 1} = \frac{-j \mu_1 wIA}{4\pi} (\frac{1}{r^2} + \frac{\gamma_1}{r}) e^{-\gamma_1 r} \sin\theta \qquad (2\text{-}17)$$

The corresponding wave function ψ_1^o which gives rise to this field is then

$$\psi_1^o = \frac{j \mu_1 wIA}{4\pi} (\frac{1}{r} + \gamma_1) e^{-\gamma_1 r} \cos\theta \qquad (2\text{-}18)$$

Here ψ_1^o is a solution obtained under the assumption that all of the medium is the same as that of the region 1. This can be written in terms of the spherical Bessel function[12] of order one of the third type as follows:

$$\psi_1^o = C k_1 (\gamma_1 r) \cos\theta \qquad (2\text{-}19)$$

where

$$C = j \mu_1 wIA \gamma_1 / 4\pi \qquad (2\text{-}20)$$

10, 11 *Ibid*.

10, 11 Stratton, *op. cit.*, Chapter 7 and 8.

12 The spherical Bessel functions of argument Z are defined by $k_1 (z) = (1 + \frac{1}{z}) e^{-z}$ and $i_1 (z) = \cosh Z - \sinh Z/Z$.

Since $\gamma_1 \neq \gamma_2$, taking the reflection of the wave into consideration, the general solution for the complete wave function ψ_1 inside the cavity, and the complete wave function ψ_2 outside the cavity can then be written, respectively, as:

$$\psi_1 = C\, k_1(\gamma_1 r)\cos\theta + A\, i_1(\gamma_1 r)\cos\theta \tag{2-21}$$

and

$$\psi_2 = B\, k_1(\gamma_2 r)\cos\theta \tag{2-22}$$

where i_1 is a spherical Bessel function of order one of the first type. The coefficient A and B can be found by applying the boundary conditions which require the continuity of tangential electric and magnetic fields at the boundary of the two regions.

The boundary conditions stated above can be represented as follows[13]:

$$N \times (E_1 - E_2) = 0 \text{ or } N\times E_1 = N\times E_2 \tag{2-23}$$

and

$$N \times (H_1 - H_2) = K \text{ or } N\times H_1 = K + N\times H_2 \tag{2-24}$$

Where N is the unit vector perpendicular to the boundary surface, and K is the surface current density at the boundary surface. In our case $K = 0$.

From (2-21), (2-22), (2-12) and (2-13) we find $E_{\phi 1}$, $E_{\phi 2}$, $H_{\theta 1}$, and $H_{\theta 2}$ then substitute them into (2-23) and (2-24) and set $r = a$ we get two equations from which A and B can be solved (In this case Hrm is not considered because N_r Hrm$= 0$.) :

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[13]Samuel Silver, Microwave Antenna Theory and Design, page 67, McGraw-Hill Book Co., Inc.: New York, 1949.

$$E_{\phi 1} = \frac{1}{r}\frac{\partial \psi_1}{\partial \theta} = \frac{1}{r}\left[Ck_1\ (\gamma_1 r) + A\ i_1\ (\gamma_1 r)\right]\sin\theta$$

$$E_{\phi 2} = \frac{1}{r}\frac{\partial \psi_2}{\partial \theta} = \frac{1}{r}\left[Bk_1\ (\gamma_2 r)\sin\theta\right]$$

$$N_r \times E_{\phi m} = E_{\phi m} N_\theta$$

therefore

$$Ck_1\ (\gamma_1 a) + A\ i_1\ (\gamma_1 a) = Bk_1\ (\gamma_2 r) \qquad (2\text{-}25)$$

With the relation $\frac{df\ (\gamma r)}{dr} = \gamma \frac{df(u)}{du}$, where $u = \gamma r$, in mind we find

$$H_{\theta 1} = \frac{1}{j\mu_1 wr}\frac{\partial^2 \psi_1}{\partial r \partial \theta}$$

$$= \frac{1}{j\mu_1 wr}\left[Ck_1'\ (\gamma_1 r) + A\ i_1'\ (\gamma_1 r)\right]\sin\theta$$

$$H_{\theta 2} = \frac{1}{j\mu_2 wr}\frac{\partial^2 \psi_2}{\partial r \partial \theta} = \frac{1}{j\mu_2 wr} B_1 k_1'\ (\gamma_1 r)\sin\theta$$

$$N_r \times H_{\theta m} = H_{\theta m} N_\phi$$

therefore

$$Ck_1'\ (\gamma_1 a) + A\ i_1'\ (\gamma_1 a) = \frac{\mu_1}{\mu_2} B_1 k_1'\ (\gamma_1 a) \qquad (2\text{-}26)$$

Setting $x = \gamma_1 a$, $y = \gamma_2 a$, and $q = \mu_1/\mu_2$, and solving (2-23) and (2-24) simultaneously a following expression for A is obtained

$$A = \frac{C\begin{vmatrix} -k_1\ (x) & -k_1\ (y) \\ -\gamma_1\ k_1'\ (x) & -q\ \gamma_2 k_1'\ (y) \end{vmatrix}}{\begin{vmatrix} i_1\ (x) & -k_1\ (y) \\ \gamma_1\ i'(x) & -q\,\gamma_2 y k_1'(y) \end{vmatrix}}$$

Multiplying the second row of each determinant with a, then setting $a\ \gamma_1 = x$ and $a\,\gamma_2 = y$, A becomes

$$A = C\,\frac{qyk_1\ (x)k_1'\ (y) - xk_1\ (y)k_1'\ (x)}{xi_1'\ (x)k_1\ (y) - qyk_1'\ (x)k_1'(y)} \qquad (2\text{-}27)$$

In a similar manner

$$B = C\,\frac{xk_1(x)i'_1(x) - k'_1(x)i_1(x)}{xi'_1(x)k_1(y) - qyk'_1(y)i_1(y)} \qquad (2\text{-}28)$$

The wave functions ψ_1 and ψ_2 which are for inside and outside of the cavity, respectively, are then completely specified.

From (2-20)

$$IA = \frac{4\pi C}{j\mu_1 w\gamma_1} \qquad (2\text{-}29)$$

Defining $(IA)_e$, the equivalent magnetic dipole contained within the cavity, in the same form as (2-29)

$$(IA)_e = \frac{4\pi B_1}{j\mu_2 w\gamma_2} \qquad (2\text{-}30)$$

By carrying out the operations indicated by (2-11), (2-12) and (2-13) the field equations in the region outside the cavity are given by

$$H_{2r} = \frac{(IA)_e}{2\pi}\left(\frac{1}{r^3} + \frac{\gamma_2}{r^2}\right)e^{-\gamma_2 r}\cos\theta \qquad (2\text{-}31)$$

$$H_{2\theta} = \frac{(IA)_e}{4\pi}\left(\frac{1}{r^3} + \frac{\gamma_2}{r^2} + \frac{\gamma_2^{\,2}}{r}\right)e^{-\gamma_2 r}\sin\theta \qquad (2\text{-}32)$$

$$E_{2\Phi} = \frac{-j\mu_2 w(IA)_e}{4\pi}\left(\frac{1}{r^2} + \frac{\gamma_2}{r}\right)e^{-\gamma_2 r}\sin\theta \qquad (2\text{-}33)$$

which are in the same forms as those of (2-15), (2-16) and (2-17).

From (2-29) and (2-30) the ratio of $(IA)_e$ to IA is given by

$$\frac{(IA)_e}{IA} = \frac{q\,\gamma_1 B_1}{\gamma_2 C} \qquad (2\text{-}34)$$

The above ratio may be simplified by considering the nature of the medium within the cavity and the limitation of the operating

frequencies. In actual applications

$$\sigma_1 = 0, \quad \mu_1 = \mu_2 = \mu_0 = 4\pi \times 10^{-7}, \quad \frac{\mu_1}{\mu_2} = 1$$

$$\gamma_1 = \sqrt{j\,\sigma_1 \mu_1 w - \epsilon_1 \mu_1 w^2} = jw\sqrt{\epsilon_1 \mu_1} = j2\pi f\sqrt{\epsilon_1 \mu_1}$$

$$|x| = |\gamma_1 a| = 2\pi fa\sqrt{\epsilon_1 \mu_1} = \frac{1}{3} \times 10^{-8} \times 2\pi fa = 2.1 \times 10^{-8} fa$$

For a 0.1 and f less than five megacycles per second, $|x|$ is less than 0.01. If $|x| \ll 1$, it can be shown[14] that the ratio of magnetic moment is

$$\frac{(IA)_e}{IA} = \frac{3e^y}{y^2+3y+3} \tag{2-35}$$

From (2-35) it is quite evident that this ratio approaches the value unity at the magnitude of $y = \gamma_2 a$ approaches zero. In other words, at sufficient low frequencies the cavity has little or no effect on observed fields in the exterior region.

Equations (2-31), (2-32) and (2-33) are field equations in the conducting medium outside the insulating cavity. Equation (2-35) is an approximate formula for effective equivalent magnetic moment $(IA)_e$. This formula is accurate enough for practical applications for frequencies less than five megacycles per second.

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[14] See Appendix B.

CHAPTER III

THE POWER RADIATED AND THE POWER DISSIPATED IN THE CONDUCTING MEDIUM

By applying Poynting's theorem to the results obtained in the previous chapter, the formulas in general form for the total outward flow of power through the insulating cavity and the total power dissipated in the conducting medium can be found.

The Total Power Crossing the Wall of the Cavity

Since the field equations (3-31), (3-32), and (3-33) obtained in Chapter II are sinusoidal time varying fields, which are represented by the complex exponential e^{jwt} as that given by equation (2-5), the average value of the outward flow of power[15] is

$$S_r = \text{Average } (E \times H)$$

$$= \frac{1}{2} R_e(E \times H^*)$$

where H^* is the complex conjugate of H.

From equations (2-32) and (2-33)

$$S_r = \frac{1}{2} R_e(E_{2\phi} \times H^*_{2\theta})$$

$$= \frac{1}{2} R_e(E_{2\phi} H^*_{2\theta})$$

$$= \frac{1}{2} R_e\left\{\left[\frac{-j\mu_2 w(IA)e}{4\pi} \left(\frac{1}{r^2} + \frac{\gamma_2}{r}\right)e^{-\gamma_2 r}\sin\theta\right] \cdot \left[\frac{(IA)e}{4\pi} \left(\frac{1}{r^3} + \frac{\gamma_2}{r^2} + \frac{\gamma_2^2}{r}\right)e^{-\gamma_2 r}\sin\theta\right]^*\right\}$$

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[15] G. P. Harnwell, Electromagnetism, Chapter 8, McGraw-Hill Book Co., Inc.: New York, 1947.

$$= \frac{\mu_2 w (IA)_e (IA)_e{*} \sin^2\theta}{32\pi^2} R_e\left\{\left[-j\left(\frac{1}{r^2} + \frac{\gamma_2}{r}\right)e^{-\gamma_2 r}\right]\right.$$

$$\left.\cdot\left[\left(\frac{1}{r^3} + \frac{\gamma_2^*}{r^2} + \frac{\gamma_2^{2*}}{r}\right)e^{-\gamma_2^* r}\right]\right\}_{16}$$

where γ_2^* is the complex conjugate of γ_2,

$$= \frac{|(IA)_e|^2 \mu_2 w \sin^2\theta}{32\pi^2 r^5} R_e\left\{\left[-j(1 + \alpha r + j\beta r)e^{-\alpha r - j\beta r}\right]\right.$$

$$\left.\left[(1 + \alpha r - j\beta r + \alpha^2 r^2 - \beta^2 r^2 - 2j\alpha\beta r^2)\,e^{-\alpha r + j\beta r}\right]\right\}$$

where $\alpha = R_e\,\gamma_2$ and $\beta = I_m\,\gamma_2$,

$$= \frac{|(IA)_e|^2 \mu_{2}w \sin^2\theta\ e^{-2\alpha r}}{32\pi^2 r^5} R_e\{[\beta r - j(1 + \alpha r)]$$

$$[1 + \alpha r + (\alpha^2 - \beta^2)r^2 - j\beta r(1 + 2\alpha r)$$

$$= \frac{|(IA)_e|^2 \mu_2 w \sin^2\theta\ e^{-2\alpha r}}{32\pi^2 r^5}\left\{\beta r^2\left[-2\alpha - (\alpha^2 + \beta^2)r\right]\right\}$$

$$= \frac{-|(IA)_e|^2 \mu_2 \beta w \sin^2\theta\ e^{-2\alpha r}}{32\pi^2}\left(\frac{2\alpha}{r^3} + \frac{|\gamma_2|^2}{r^2}\right) \text{ Watts/m}^2$$

The total outward flow of power, Pr, through the wall of the insulating cavity of radius a is the closed integral of $S_r ds$ over the surface of the cavity and is given by

$$P_r = \int_s S_r ds$$

The element of area ds in spherical coordinates is given by

$$ds = a \sin\theta\, d\theta\, d\phi$$

---

16 $(Z_1 + Z_2 - \cdot \cdot \cdot + Z_n)^* = Z_1^* + Z_2^* + \cdot \cdot \cdot \cdot + Z_n^*$, $Z Z^* = |Z|^2$ $(Z_1 Z_2 \cdot \cdot \cdot Z_n)^* = Z_1^* Z_2^* \cdot \cdot Z_n^*$, $Z + Z^* = 2 R_e Z$

Then

$$P_r = \frac{-|(IA)_e|^2 \ \mu_2 \ \beta w e^{-2\alpha a}}{32 \pi^2} \left(\frac{2\alpha}{a} + |\gamma_2|^2 \right) \int_0^{2} d\phi \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{-|(IA)_e|^2 \ \mu_2 \ \beta w e^{-2\alpha a}}{16 \pi} \left(\frac{2\alpha}{a} + |\gamma_2|^2 \right) \int_0^{\pi} \sin^3\theta \, d\theta$$

where

$$\int_0^{\pi} \sin^3\theta \, d\theta = \left[-\frac{1}{3} \cos\theta (\sin^2\theta + 2) \right]_0^{\pi} = \frac{4}{3}$$

therefore

$$P_r = \frac{-|(IA)_e|^2 \ \mu_2 \ \beta w e^{-2\alpha a}}{12 \pi} \left(\frac{2\alpha}{a} + |\gamma_2|^2 \right) \tag{3-1}$$

From (2-33)

$$(IA)_e = \frac{3 e^{y} IA}{y^2 + 3y + 3}, \quad y = \gamma_2 a$$

then

$$|(IA)_e|^2 = (IA)_e (IA)_e^*$$

$$= \frac{9 |IA|^2 e^{2\alpha a}}{|a^2 \gamma_2^2 + 3a \gamma_2 + 3|^2} \tag{3-2}$$

Substituting (3-2) into (3-1)

$$P_r = \frac{3 |IA|^2 \mu_2 \beta w}{4\pi |a^2 \gamma_2^2 + 3a \gamma_2 + 3|^2} \left(\frac{2\alpha}{a} + |\gamma_2|^2 \right) \text{ Watts} \tag{3-3}$$

Equation (3-3) is the general expression of the total outward flow of power through the entire wall of the cavity. P_r is a constant which depends upon the nature of the conducting medium, the frequency, the radius of the insulating cavity, and the equivalent magnetic moment of the loop antenna.

The Total Power Dissipated in the Conducting Medium

The power dissipated per unit volume in a conducting medium is always given by[17] $E \cdot \dot{i}$, where $\dot{i}$ represents the current density vector in the conducting medium. In this case $\dot{i}$ is everywhere in the same direction as E, therefore

$$E \cdot \dot{i} = E\dot{i} \tag{3-4}$$

$\dot{i}$ can be shown to be in the same direction as E by using Maxwell's equation. In this case

$$E = E_2 = E_{2\phi}$$

$$H = H_2 = H_{2r} + H_{2\theta}$$

and

$$E_{2r} = E_{2\theta} = H_{2\phi} = 0$$

$$H = \dot{i} + \epsilon \frac{\partial E}{\partial t} \tag{3-5}$$

This equation, which is one of Maxwell's equations, shows $\dot{i}$ is in the direction of H. In spherical polar coordinates system

$$H_2 = \begin{vmatrix} \dfrac{N_r}{r^2 \sin\theta} & \dfrac{N_\theta}{r \sin\theta} & \dfrac{N_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ H_{2r} & rH_{2\theta} & 0 \end{vmatrix} \tag{3-6}$$

Since H_{2r} and $H_{2\theta}$ are functions of r and θ only as shown by equations (2-29) and (2-30), the expansion of the right member of equation

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[17] Edward C. Jordan, Electromagnetic Waves and Radiating Systems, Chapter 6, Prentice-Hall, Inc., 1960.

(3-6) contains only one non-zero component. The non-zero component is in the ϕ direction which is the same as the direction of $E_{2\phi}$. In (3-6) N_r, N_θ, and N_ϕ are unit vectors in r, θ, and ϕ directions respectively.

From Ohm's law, which can be applied in this case,

$$i = \sigma E$$

The power dissipated per unit volume, P, becomes

$$P = E \cdot i = \sigma E^2$$

For sinusoidal time varying fields which are represented by the exponential factor e^{jwt}, the average power dissipation per unit volume is given by

$$P = \frac{1}{2} E \cdot i^*$$

$$= \frac{1}{2} \sigma E E^* \qquad (3\text{-}7)$$

where the * indicates the complex conjugate.

From the field equation of E (2-31)

$$P = \frac{\sigma_2}{2} E_{2\phi} \cdot E_{2\phi}^*$$

$$= \frac{\sigma_2}{2}\left[\frac{-j \mu_2 w (IA)_e}{4\pi} \left(\frac{1}{r^2} + \frac{\gamma_2}{r}\right) e^{-\gamma_2 r} \sin\theta\right]$$

$$\cdot \left[\frac{-j \mu_2 w (IA)_e^*}{4\pi} \left(\frac{1}{r^2} + \frac{\gamma_2^*}{r}\right) e^{-\gamma_2^* r} \sin\theta\right]$$

$$= \frac{\sigma_2 |(IA)_e|^2 \mu_2^2 w^2}{32 \pi^2 r^4} (1 + \gamma_2 r)(1 + \gamma_2^* r) e^{-2\alpha r} \sin^2\theta$$

$$= \frac{\sigma_2}{2}\left[\frac{|(IA)_e| \mu_2 w}{4\pi r^2}\right]^2 \left[1 + (\gamma_2 + \gamma_2^*) r + \gamma_2 \gamma_2^* r^2\right] e^{-2\alpha r} \cdot \sin^2\theta$$

$$= \frac{\sigma_2}{2}\left[\frac{|(IA)_e|\ \mu_2 w}{4\pi r^2}\right]^2 (1 + 2\alpha r + |\gamma_2|^2 r^2)\, e^{-2\alpha r} \sin^2\theta$$

or

$$P = \frac{\sigma_2}{2}\left[\frac{|(IA)_e|\,\mu_2 w}{4\pi}\right]^2 \left(\frac{1}{r^4} + \frac{2\alpha}{r^3} + \frac{|\gamma_2|^2}{r^2}\right) e^{-2\alpha r} \sin^2\theta$$

$$\text{Watts/m}^3 \qquad (3\text{-}8)$$

The average power dissipated in a volume enclosed between the insulating cavity and a concentric sphere of radius r, where $r > a$, is

$$P_{d1} = \int_v P_{dv}$$

In spherical coordinates

$$dv = r^2 \sin^2\theta \ dr\ d\theta\ \ d\phi$$

Then

$$P_{d1} = \frac{\sigma_2}{2}\left[\frac{|(IA)_e|\,\mu_2 w}{4\pi}\right]^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta\ d\theta \int_a^r \left(\frac{1}{r^2} + \frac{2\alpha}{r} + |\gamma_2|^2\right) \cdot e^{-2\alpha r} dr$$

$$= \frac{\sigma_2}{2}\left[\frac{|(IA)_e|\,\mu_2 w}{4\pi}\right]^2 2\pi \frac{4}{3} \int_a^r \left(\frac{1}{r^2} + \frac{2\alpha}{r} + |\gamma_2|^2\right) e^{-2\alpha r} dr$$

Since

$$\int_a^r \left(\frac{1}{r^2} + \frac{2\alpha}{r} + |\gamma_2|^2\right) e^{-2\alpha r} dr$$

$$= \int_a^r \left(\frac{2\alpha r\, e^{-2\alpha r} + e^{-2\alpha r}}{r^2}\right) dr + \int_a^r |\gamma_2|^2 e^{-2\alpha r} dr$$

$$= \left[-\frac{e^{-2\alpha r}}{r} + |\gamma_2|^2 \frac{e^{-2\alpha r}}{-2\alpha}\right]_a^r$$

therefore

$$P_{d1} = \frac{\sigma_2\, |(IA)_e|^2\, \mu_2^2\, w^2}{12\pi} \left[\frac{e^{-2\alpha r}}{r} + |\gamma_2|^2 \frac{e^{-2\alpha r}}{2\alpha}\right]_r^a$$

$$= \frac{\sigma_2 [|(IA)_e| \mu_2 w]^2}{12\pi} [(\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha})e^{-2\alpha a} -$$

$$(\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha})e^{-2\alpha r}] \text{ Watts} \qquad (3\text{-}9)$$

By letting r approach infinity in the limit, the total average power dissipated in the conducting medium is

$$P_d = \lim_{r \to \infty} P_{d1}$$

$$= \frac{\sigma_2 |(IA)_e|^2 \mu_2^2 w^2}{12\pi} (\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha})e^{-2\alpha a} \qquad (3\text{-}10)$$

Substituting (3-2) into (3-10)

$$P_d = \frac{\sigma_2 \mu_2^2 w^2}{12\pi} (\frac{9|IA|^2}{|a^2\gamma_2^2 + 3a\gamma_2 + 3|^2}) (\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}) \qquad (3\text{-}11)$$

or

$$P_d = \frac{3|IA|^2 \sigma_2 \mu_2^2 w^2}{4\pi|a^2\gamma_2^2 + 3a\gamma_2 + 3|^2} (\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}) \text{ Watts} \qquad (3\text{-}12)$$

Since the conducting medium outside the cavity is assumed to be infinite in extent, then according to the conservation of energy, the total outward flow of power, P_r, should be equal to the total power dissipated, P_d, in the conducting medium. The fact that $P_d = P_r$ can be shown as follows:

From (3-12) and (3-3)

$$P_d = \frac{3|IA|^2 \mu_2 \beta w}{4\pi|a^2\gamma_2^2 + 3a\gamma_2 + 3|^2} (\frac{\sigma_2 \mu_2 w}{\beta}) \frac{1}{2\alpha} (\frac{2\alpha}{a} + |\gamma_2|^2)$$

$$= (\frac{\sigma_2 \mu_2 w}{2\alpha\beta}) P_r$$

The coefficient $\frac{\sigma_2 \mu_2 w}{2\alpha\beta}$ is unity. From equations (2-8) and (2-9)

$$\frac{\sigma_2 \mu_2 w}{2\alpha\beta} = \frac{\sigma_2 \mu_2 w}{2\left[w\sqrt{\frac{\mu_2 \epsilon_2}{2}\left(\sqrt{1+\frac{\sigma_2^2}{w^2 \epsilon_2^2}}-1\right)}\right]\left[w\sqrt{\frac{\mu_2 \epsilon_2}{2}\left(\sqrt{1+\frac{\sigma_2^2}{w^2 \epsilon_2^2}}+1\right)}\right]}$$

$$= \frac{\sigma_2 \mu_2 w}{2 w^2 \frac{\mu_2 \epsilon_2}{2}\sqrt{1+\frac{\sigma_2^2}{w^2 \epsilon_2^2}-1}}$$

$$= 1$$

Therefore

$$P_d = P_r$$

$$= \frac{3|IA|^2 \mu_2 \beta w}{4\pi \left| a^2 \gamma_2^2 + 3a\gamma_2 + 3\right|^2}\left(\frac{2\alpha}{a} + |\gamma_2|^2\right) \tag{3-13}$$

In this chapter equations for P_d and P_r have been derived. It has been shown that P_d and P_r are equal satisfying the requirement of the conservation of energy.

CHAPTER IV

DISCUSSION OF RESULTS

In this chapter the physical significance of each component of the field equations is discussed. Formulas for the electromotive forces induced in a small receiving loop antenna placed at different positions are derived and discussed. The power dissipation vs. radial distance curves for sea water at different frequencies are shown in Figure 4. Figure 5 shows the same type of curves at a fixed frequency with the radius of the insulating cavity varied.

Discussion of Field Equations in the Conducting Medium

The field equations in the conducting medium obtained in Chapter II are

$$H_{2r} = \frac{(IA)_e}{2\pi}\left(\frac{1}{r^3} + \frac{\gamma_2}{r^2}\right)e^{-\gamma_2 r}\cos\theta \qquad (2\text{-}31)$$

$$H_{2\theta} = \frac{(IA)_e}{2\pi}\left(\frac{1}{r^3} + \frac{\gamma_2}{r^2} + \frac{\gamma_2^{\,2}}{r}\right)e^{-\gamma_2 r}\sin\theta \qquad (2\text{-}32)$$

and

$$E_{2\phi} = \frac{-j\,\mu_2\,w\,(IA)_e}{4\pi}\left(\frac{1}{r^2} + \frac{\gamma_2}{r}\right)e^{-\gamma_2 r}\sin\theta \qquad (2\text{-}33)$$

where H_{2r}, $H_{2\theta}$, and $E_{2\phi}$ are three component fields. The sinusoidal time variation which is represented by the real part of the complex exponential e^{jwt} is included in the coefficient $(IA)_e$ in every field equation. All component fields attenuate exponentially in accordance with the attenuation factor $e^{-(R_e\,\gamma_2)r}$ which is symbolized by $e^{-\alpha r}$.

These fields can be resolved into three kinds of partial fields according to their dependence upon the radial distance r:

(1) the "static field" varying inversely with r^3,

(2) the "induction field" varying inversely with r^2, and

(3) the "radiation field" varying inversely with r.

The components of static and induction fields are the same forms as those which would be computed from a static dipole with fixed moment. The induction field is the quasi-stationary-state field commonly observed in the neighborhood of a circuit element at low frequencies; the magnetic component of the induction field is that which would be calculated on the basis of the Biot-Savart law for stationary currents. At small distances from the dipole the static and induction fields predominate. At a distance where

$$r > \frac{\lambda}{2\pi}$$

the radiation field becomes the predominant term and the static and induction fields become negligible.

Electromotive Force Induced in a Small Loop Antenna

If a small insulated receiving loop antenna with its area small enough so that the magnetic flux density can be considered uniform in the area, and if the axis of the loop is in r direction, then the electromotive force induced[18] in the loop is given by

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[18]Induced electromotive force $= -\frac{d\Phi}{dt} = \frac{-dB}{dt}\Delta s = -\mu\frac{dH}{dt}\Delta s$ where Φ = magnetic flux, and B = magnetic flux density.

$$e = -\mu_2 \frac{dH_{2r}}{dt} \Delta s$$

$$= -jw\mu_2 \Delta s H_{2r}$$

or

$$e = \frac{-(IA)_e j \mu_2 w \Delta s}{2\pi} \left(\frac{1}{r^3} + \frac{\gamma_2}{r^2}\right) e^{-\gamma_2 r} \cos\theta \quad \text{volts} \qquad (4\text{-}1)$$

where Δs is the area of the receiving loop. The maximum induced electromotive force occurs at $\theta = 0$ and $\theta = \pi$.

If the axis of the receiving loop is in θ direction, then the induced electromotive force is

$$e = -\mu_2 \frac{dH_2}{dt} \Delta s$$

or

$$e = \frac{-(IA)_e j \mu_2 w \Delta s}{2\pi} \left(\frac{1}{r^3} + \frac{\gamma_2}{r^2} + \frac{\gamma_2^2}{r}\right) e^{-\gamma_2 r} \sin\theta \quad \text{volts} \qquad (4\text{-}2)$$

In this case the maximum induced electromotive force occurs at $\theta = \pi/2$.

When the receiving loop is placed near to the transmitting loop the radiation component of equation (4-2) can be neglected, then the maximum induced electromotive force at $\theta = 0$, $\theta = \pi$, and $\theta = \pi/2$ for both cases are the same. If the distance is greater than $\lambda/2\pi$ the radiation component in equation (4-2) is predominant and the maximum induced electromotive force occurs only at $\theta = \pi/2$, which is

$$e_m = \frac{-(IA)_e j \mu_2 w \Delta s}{2\pi r} e^{-\gamma_2 r} \quad \text{volts} \qquad (4\text{-}3)$$

<u>Discussion of Power Dissipation and Power Radiation</u>

Several expressions for power were derived in Chapter III, the average power dissipated in a volume confined by the insulating cavity

and a concentric sphere of radius r is given by

$$P_{d1} = \frac{\sigma_2 \left(|(IA)_e| \; \mu_2 w\right)^2}{12\pi} \left[\left(\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}\right) e^{-2\alpha a} - \left(\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha}\right) e^{-2\alpha r} \right] \quad \text{Watts} \quad (3\text{-}9)$$

or

$$P_{d1} = \frac{3\,|IA|^2 \, \sigma_2 \, \mu_2^2 \, w^2}{4\pi \, |a^2 \gamma_2^2 + 3a\gamma_2 + 3|^2} \left[\left(\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}\right) e^{-2\alpha a} - \left(\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha}\right) e^{-2\alpha r} \right] \quad \text{Watts} \quad (3\text{-}9\text{-}a)$$

The total average power dissipated in the conducting medium, P_d, and the total average power radiated through the insulating cavity, P_r, are equal in magnitude and were found to be

$$P_r = P_d = \frac{3\,|IA|^2 \, \mu_2 \beta w}{4\pi \, |a^2 \gamma_2^2 + 3a\gamma_2 + 3|^2} \left(\frac{2\alpha}{a} + |\gamma_2|^2\right) \quad \text{Watts} \quad (3\text{-}13)$$

or

$$P_r = P_d = \frac{3\,|IA|^2 \, \sigma_2 \, \mu_2^2 \, w^2}{4\pi \, |a^2 \gamma_2^2 + 3a\gamma_2 + 3|^2} \left(\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}\right) \quad \text{Watts} \quad (3\text{-}13\text{-}a)$$

In finding P_r the effect of H_{2r} was not considered. The average power flow per unit area in θ direction is represented by $\frac{1}{2}$ Re $(E_{2\phi} H_{2r}^*)$. It is interesting to note that the total average power flow in the θ direction, P_θ, is always equal to zero and is shown as follows:

$$P_\theta = \frac{1}{2} \int_s \mathrm{Re}\,(E_{2\phi} \, H_{2r}^*)\, ds$$

where $ds = r \sin\theta \, d\phi \, dr$, then

$$P_\theta = \frac{-|(IA)_e|^2 \ \mu_2 w}{8\pi^2} \int_0^{2\pi} d\phi \int_0^{\pi} \cos\theta \ \sin^2\theta \ d\theta$$

$$\cdot \int_a^{\infty} \left(\frac{1}{r^2} + \frac{\gamma_2}{r}\right)\left(\frac{1}{r^3} + \frac{\gamma_2^*}{r^2}\right) e^{-2\alpha r} \, dr$$

but since

$$\int_0^{\pi} \cos\theta \ \sin^2\theta \ d\theta = \int_0^{\pi} \cos\theta \ d\theta - \int_0^{\pi} \cos^3\theta \ d\theta$$

$$= \left[\sin\theta - \frac{1}{3}\sin\theta \ (\cos^2\theta + 2)\right]_0^{\pi}$$

$$= 0$$

then

$$P_\theta = 0$$

In Chapter III it was proved that P_r is equal to P_d. They should be equal according to the universal low of conservation of energy which was applied as a check on the validity of equations for P_r and P_d.

The equation for P_{d1} is important in practical applications. It can be written as

$$P_{d1} = \frac{\sigma_2 \ |(IA)_e|^2 \ \mu_2^2 \ w^2}{12\pi}\left[\left(\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}\right)e^{-2\alpha a} - \left(\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha}\right)e^{-2\alpha r}\right]$$

$$= P_d - \frac{\sigma_2 |(IA)_e|^2 \ \mu_2^2 \ w^2}{12\pi}\left(\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha}\right)e^{-2\alpha r} \qquad (4\text{-}4)$$

or

$$P_{d1} = P_d - \frac{3\,|IA|^2 \ \sigma_2 \ \mu_2^2 \ w^2}{4\pi\,|a^2\gamma_2^2 + 3a\gamma_2 + 3|^2}\left(\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha}\right)e^{-2\alpha r} \qquad (4\text{-}5)$$

From which

$$\frac{P_{d1}}{P_d} = 1 - \left(\frac{\frac{1}{r} + \frac{|\gamma_2|^2}{2\alpha}}{\frac{1}{a} + \frac{|\gamma_2|^2}{2\alpha}} \right) e^{-2\alpha(r-a)} \tag{4-6}$$

The second term on the right hand side of equation (4-4) or (4-5) represents the power dissipated outside a concentric sphere of radius r.

Figure 4 shows an example when the conducting medium is sea water[19] with $\sigma_2 = 4$, $\mu_2 = \mu_0 = 1.2566 \times 10^{-6}$, and $\epsilon_r = \epsilon_2/\epsilon_0 = 81$. The radius of the insulating cavity is 0.1 meter, and the equivalent magnetic dipole moment of the loop antenna is assumed to be 10. This figure shows P_{d1}/P_d vs. r curves for the frequencies of 5 megacycles, 500 kilocycles, 100 kilocycles and 50 kilocycles. The value of the total average dissipated power, P_d, and the value of the attenuation constant, α, for each curve was calculated and is shown in Table II.

TABLE II. TABLE OF P_d AND α

f	P_d (Watts)	α
50 kilocycles	12.6	0.89
100 kilocycles	46.4	1.256
500 kilocycles	698	2.81
5 megacycles	2400	8.88

[19]Hugh Hildreth Skilling, Fundamentals of Electric Waves, Second Ed., pp. 184, John Wiley and Sons, Inc., New York, 1948.

As can be observed from Table II and from the curves shown in Figure 4, the total power dissipated in the conducting medium, or the total power radiated through the insulating cavity, and the attenuation constant both increase with frequency. The higher the frequency the greater the power radiated, but since α also increases with frequency, most of the total radiated power at high frequencies dissipated as heat in the vicinity of the radiating system. For the case of sea water, because of relatively high conductivity the radiated power attenuates very rapidly.

Figure 5 shows P_{d1}/P_d vs. r curves for sea water at three different radii of insulating cavity at a fixed frequency of 500 kilocycles. All other constants are the same as that of Figure 4. When the radius of the insulating cavity becomes smaller a similar effect happens as when the frequency increases in Figure 4.

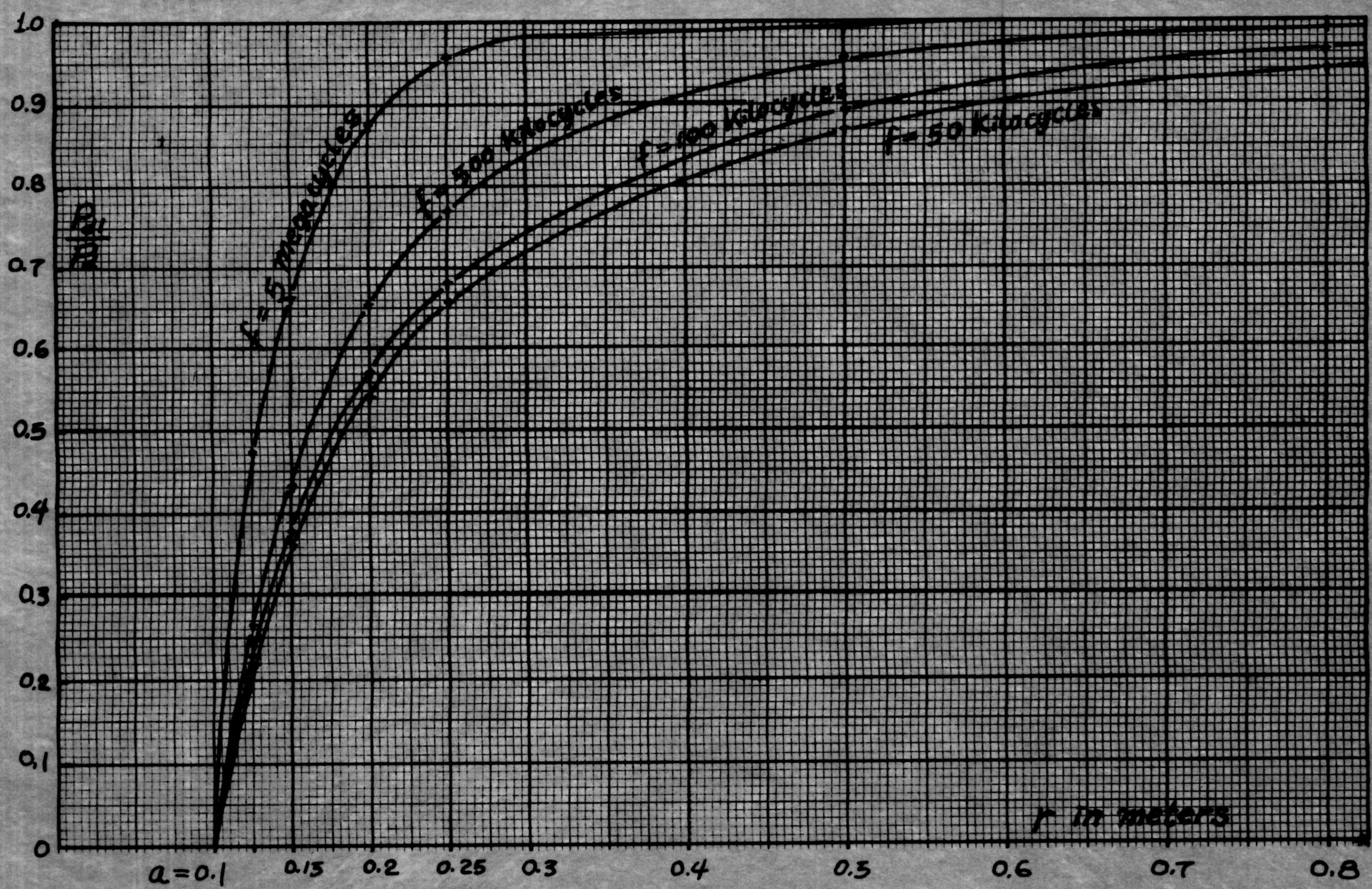

Figure 4. P_{d1}/P_d vs. r Curves for Sea Water at Four Different Frequencies.
$|IA| = 10$, $a = 0.1$, $\sigma_2 = 4$, $\epsilon_r = 81$, $\mu_2 = \mu_0$.

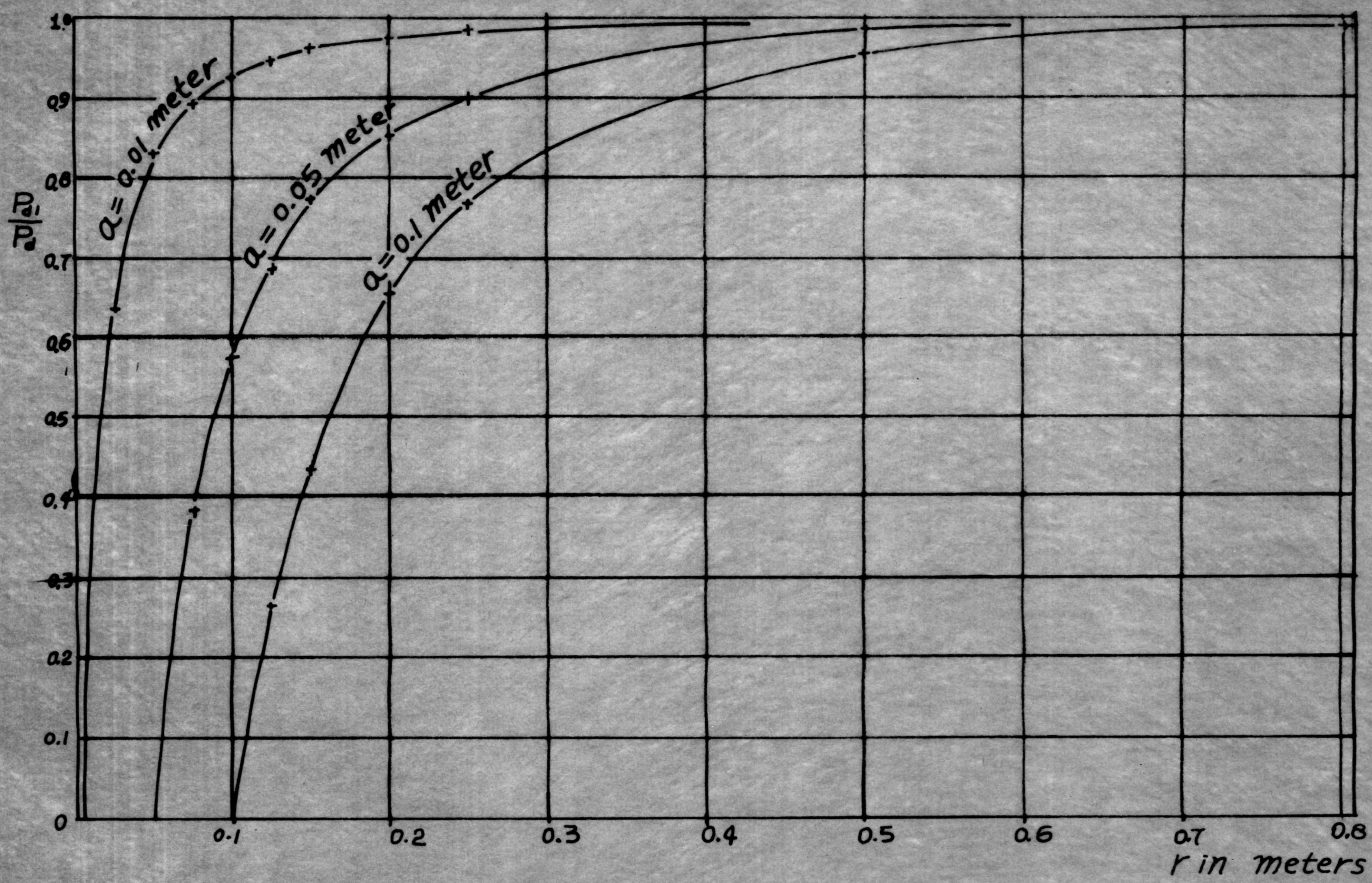

Figure 5. P_{d1}/P_d vs. r Curves for Sea Water at Three Different Radii of the Cavity.

$|IA| = 10$, $\sigma_2 = 4$, $\epsilon_r = 81$, $\mu_2 = \mu_0$, $f = 500$ Kilocycles.

Conclusion

The nature of the electromagnetic wave in a conducting medium generated by a loop antenna has been studied. Based on the formulas derived, a low frequency wave suffers less attenuation than a high frequency wave. This effect is very noticeable when the conductivity of the medium is high, such as with sea water. At low frequencies the power radiation is small compared with the power supplied to the loop antenna. The selection of frequency and power of a transmitter depends upon the conductivity of the medium, the size of the loop antenna, and the size of the insulating cavity. However, high frequencies are inefficient, because at these frequencies most of the power is dissipated as heat in the vicinity of the antenna. Furthermore, the frequency cannot be too low because of a great amount of power required. In addition, size of a low frequency transmitter is generally larger than a high frequency transmitter. In some practical applications size is very important and consequently larger transmitters cannot be used.

APPENDIX

APPENDIX A

In terms of the primary constants of the medium σ, μ, and ϵ, the values of α are

$$\alpha = \mathrm{Re}\,\gamma$$

$$= \mathrm{Re}\sqrt{jw\mu(\sigma + jw\epsilon)}$$

$$= \mathrm{Re}\left[\sqrt{w\mu(w^2\epsilon^2 + \sigma^2)^{1/2}}\right] e^{j\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{-w\epsilon}\right)}$$

$$= \left[\sqrt{w\mu(w^2\epsilon^2 + \sigma^2)^{1/2}}\right]\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{-w\epsilon}\right)\right]$$

$$= \sqrt{w\mu(w^2\epsilon^2 + \sigma^2)^{1/2}}\left[\sqrt{\frac{(w^2\epsilon^2 + \sigma^2)^{1/2} - w\epsilon}{2(w^2\epsilon^2 + \sigma^2)^{1/2}}}\right]_{20}$$

$$= \sqrt{\frac{w^2\mu\epsilon}{2}\left[\left(1 + \frac{\sigma^2}{w^2\epsilon^2}\right)^{1/2} - 1\right]}$$

$$= w\sqrt{\frac{\mu\epsilon}{2}\left[\left(1 + \frac{\sigma^2}{w^2\epsilon^2}\right)^{1/2} - 1\right]}$$

Similarly

$$\beta = \mathrm{Im}\,\gamma$$

$$= \left[\sqrt{w\mu(w^2\epsilon^2 + \sigma^2)^{1/2}}\right]\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{-w\epsilon}\right)\right]$$

$$= w\sqrt{\frac{\mu\epsilon}{2}\left[\left(1 + \frac{\sigma^2}{w^2\epsilon^2}\right)^{1/2} + 1\right.}$$

As shown in the above equations, α , the real part of the

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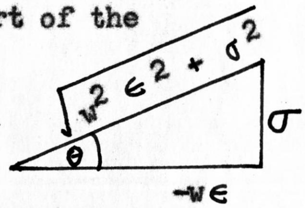
[20] $\cos = \dfrac{-w\epsilon}{\sqrt{w^2\epsilon^2 + \sigma^2}}$, $= \tan^{-1}\left(\dfrac{\sigma}{-w\epsilon}\right)$



$$\cos\frac{\theta}{2} = \sqrt{\frac{1 + \mathrm{los}\,\theta}{2}} = \sqrt{\frac{(w^2\epsilon^2 + \sigma^2)^{1/2} - w\epsilon}{2(w^2\epsilon^2 + \sigma^2)^{1/2}}}$$

propagation constant γ, is a measure of the rate at which the wave is attenuated as it progresses through the medium. β, the imaginary part of γ, is the phase shift per unit distance for the wave.

APPENDIX B

The ratio of the effective equivalent magnetic dipole moment, $(IA)_e$, to the actual equivalent magnetic dipole moment, IA, can be simplified as follows, if the defined quantity $x = \gamma_1 a$ is much less than unity.

From (2-26) and (2-32)

$$\frac{(IA)_e}{IA} = \frac{q\ \gamma_1 B}{\gamma_2 C} = (\frac{x}{yC})B$$

$$= \frac{x^2 k_1\ (x)\ i_1\ (x) - x^2 k_1'(x)\ i_1\ (x)}{xy i_1'\ (x) k_1\ (y)\ - y^2 k_1'(y)\ i_1\ (x)} \qquad \text{(B-1)}$$

where $x = \gamma_1 a$, $y = \gamma_2 a$, and $q = \frac{\mu_1}{\mu_2} = 1$. x and y are complex quantities, and

$$k_1\ (z) = (1 + \frac{1}{z})\ e^{-z} \qquad \text{(B-2)}$$

$$k_1'\ (z) = \frac{d\ k_1\ (z)}{d\ z} = -\ (\ \frac{1}{z^2} + \frac{1}{z} + 1)\ e^{-z} \qquad \text{(B-3)}$$

$$i_1\ (z) = \cosh z - \frac{\sinh z}{z} \qquad \text{(B-4)}$$

$$i_1'(z) = \frac{di_1\ (z)}{d\ z} = \sinh z - \frac{x \cosh z - \sinh z}{z^2} \qquad \text{(B-5)}$$

where z represents x or y.

Divide the denominator and the numerator of (B-1) with $i_1(x)$ and let x approach to zero

$$\lim_{x\to 0} \frac{(IA)_e}{IA} = \lim_{x\to 0} \left[\frac{x^2 k_1\ (x)\ i_1'\ (x)/i_1\ (x) - x^2 k_1'\ (x)}{xy i_1'\ (x)\ k_1\ (y)/i_1\ (x) - y^2 k_1'\ (y)} \right] \qquad \text{(B-6)}$$

In (B-6) by applying L'Hospital's rule

$$\lim_{x\to 0}\left[x^2k_1\ (x)\,\frac{i_1{}'(x)}{i_1\ (x)}\right]$$

$$=\lim_{x\to 0}\left[(x+1)e^{-x}\right]\lim_{x\to 0}\left[\frac{x^2\ \sinh x - x\cosh x + \sinh x}{x\cosh x - \sinh x}\right]\left(=\frac{0}{0}\right)$$

$$= 1\cdot\lim_{x\ \ 0}\left[\frac{\sinh x + x\cosh x}{\sinh x}\right]$$

$$=\lim_{x\to 0}\left[1+\frac{x\cosh x}{\sinh x}\right]$$

$$= 1+\lim_{x\to 0}\left[\frac{\cosh x}{\frac{\sinh x}{x}}\right] = 1+\frac{1}{1} = 2 \qquad \text{(B-7)}$$

$$\lim_{x\to 0}\left[x^2k_1{}'\ (x)\right] = \lim_{x\to 0}\left[-(1+x+x^2)e^{-x}\right] = -1 \qquad \text{(B-8)}$$

and

$$\lim_{x\to 0}\left[\frac{xi_1{}'(x)}{i_1\ (x)}\right] = \lim_{x\to 0}\left[\frac{x^2\ \sinh x - x\cosh x + \sinh x}{x\cosh x - \sinh x}\right]\left(=\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[1+\frac{x\cosh x}{\sinh x}\right] = 2 \qquad \text{(B-9)}$$

Substitute (B-7), (B-8), and (B-9) in (B-6)

$$\frac{(IA)_e}{IA} = \frac{2-(-1)}{2yk_1\ (y) - y^2k_1{}'\ (y)}$$

$$= \frac{3}{2y\ (1+\frac{1}{y})e^{-y} - y^2\left[-(1+\frac{1}{y}+\frac{1}{y^2})e^{-y}\right]}$$

$$= \frac{3\ e^y}{y^2+3y+3} \qquad \text{(3-35)}$$

This is the approximate equation for $(IA)_e/IA$ which is accurate for practical applications.

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