Elastic Stress Distribution in Rectangularly Notched Members by Finite Differences

Paul Andrew Hoffman

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ELASTIC STRESS DISTRIBUTION IN RECTANGULARLY NOTCHED MEMBERS BY FINITE DIFFERENCES

BY

PAUL ANDREW HOFFMAN

A thesis submitted in partial fulfillment of the requirements for the degree Master of Science, Major in Civil Engineering, South Dakota State University

1965
ELASTIC STRESS DISTRIBUTION IN RECTANGULARLY NOTCHED MEMBERS BY FINITE DIFFERENCES

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Sept. 28, 1964
Thesis Adviser

Sept. 28, 1964
Head, Civil Engineering Department
ACKNOWLEDGMENTS

To Dr. Zaher Shoukry, Associate Professor, Department of Civil Engineering, the author wishes to express sincere appreciation for his great assistance and encouragement in developing and analyzing the course of this research project. Dr. Shoukry's continued interest and help are greatly appreciated.

This thesis is dedicated to Donna, my wife, for her assistance and sacrifice throughout the course of the Master of Science program.

PAH
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**Plate A**

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CHAPTER I

INTRODUCTION

General

The design of notched members by structural engineers has become a practice that is now being employed more frequently than in the past. This is particularly true in the aircraft, missile, automobile and shipbuilding industries where available space is necessarily a minimum. Some form of notching is frequently incorporated into the design of a structural member to allow the passage of electrical wire bundles, hydraulic and air conduits or other structural members. Frequently a notch or cutout serves as a lightening measure or as an inspection access opening. Notched members are also finding an increased application in the building industries for various structural, space utilization and aesthetic reasons.

The determination of the elastic stress distribution in notched structural members is important because the introduction of a notch generally has the effect of subjecting the member to a situation of high combined stresses. This is especially true in the notch vicinity, even for very simple loading conditions.

An analysis of the elastic stress distribution in structural members is of particular importance because it is the first step in the investigation of the critical buckling loads.
Object and Scope of Investigation

The primary objective of this investigation is to obtain elastic stress distributions in rectangularly notched structural members, employing the finite difference method and electronic computation. The objective also includes principles of photoelasticity in order to substantiate some aspects of the numerical analysis. The solutions presented in this work will provide a valuable source of information for future investigators who seek solutions to similar problems.

The structural members investigated are either plates or beams with rectangular notches. However, the methods of analysis developed in this work apply predominantly to thin plates.

The stress distribution analysis consists of computing the normal stresses acting in the X and Y directions, the shearing stress acting on a plane normal to the X-axis and having its direction parallel to the Y-axis, the major and minor principal stresses and their orientations with respect to the X-axis. The above mentioned quantities were computed at individual points throughout the section for the separate conditions of pure compression and pure moment.

The finite difference computations were carried out with the aid of an electronic computer programmed in the Fortran II language. The computer output was programmed for an accounting machine in order that the results may be shown on the actual plan shape of the particular structural section under consideration. This permitted
stress contours to be drawn within the outlines of the section. Some verification of these stress contours was attempted by obtaining photographs of a photoelastic model subject to a condition of pure moment.

Previous Investigations

Many reports are available which discuss the elastic behavior of various structural sections containing cutouts or access holes of different shapes; \([21], [22]\) and \([23]\). There has also been considerable work investigating the effect of inelastic action on the resistance of structural members with cutouts, subjected to different types of loads \([24]\). Some investigation of stresses at web cutouts has also been accomplished by photoelastic methods \([25]\).

The dimensions of the example problems treated in this work are identical to one investigated, only for a pure moment condition, by Pyka.\(^1\) Pyka employed the method of "successive relaxation" of joint forces where the surface of the section was replaced by a \(5 \times 5\) grid system. The maximum principal stresses were calculated statistically once the joint forces had the required equilibrium. A photoelastic analysis indicated that the analytical and photoelastic results agreed within about twenty percent.

\(^*\)Numbers in square brackets refer to bibliography entries.

Shoukry\textsuperscript{2} investigated the elastic stress distribution in and buckling characteristics of the webs of castellated steel beams using the finite differences and electronic computations. The results indicated that the elastic stress distribution was accurate for the relatively fine mesh employed (18 x 10).

Several previous investigators employed the method of finite differences successfully when investigating stress distributions in deep beams.\textsuperscript{3,4,5} They concluded that the results are excellent as long as a relatively fine mesh is employed.

The results of the deep beam stress distribution analysis mentioned above indicated that the agreement between the finite difference method and a more accurate stress function solution in a closed form is good, even for a relatively coarse mesh.\textsuperscript{5} In the above case the error involved for an 8 x 8 mesh was six to seven percent when computing the normal stresses $\sigma_x$ and $\sigma_y$. This may be reduced


to approximately two percent if a finer mesh \((24 \times 24)\) is used. The error in the shearing stresses, computed by finite differences, is somewhat larger.

---

CHAPTER II

ELASTIC STRESS ANALYSIS

Choice of Analytical Method

In the stress distribution analysis of a thin plate, where the dimensions of the plate are given in Cartesian Coordinates, the ideal method of solution is to obtain an exact evaluation for the function $\phi$ which satisfies the biharmonic equation:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (1)$$

where $\phi$ is the stress function. At any point on the plate, the stress function value can be thought of as an ordinate originating at, and perpendicular to, the plate surface and extending parallel to an imaginary $\phi$-axis of the plate. When connected by a series of lines at the top, the stress function ordinates can be said to form a smooth, undulating surface above or below the plane of the plate depending on whether the ordinates are positive or negative.

If equation (1) is satisfied for all points within the outlines of the plate, the stresses can be computed from the stress function according to the equations:

---


\[ \sigma_x = \frac{\partial^2 \phi}{\partial y^2} \] (2)

\[ \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \] (3)

and

\[ \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \] (4)

where

\[ \sigma_x = \text{stress in the X direction.} \]

\[ \sigma_y = \text{stress in the Y direction.} \]

\[ \tau_{xy} = \text{shearing stress at any point.} \]

Structural sections investigated by the above method are referred to as "boundary value problems" because the boundary values of \( \phi \) can be calculated from loading conditions applied at the boundary of the section. As seen above, the solution of boundary value problems employing the stress function requires the solution of partial differential equations, namely the biharmonic equation (1) and the stress equations (2), (3) and (4). "Only in the case of simple boundaries can these equations be treated in a rigorous manner."\(^9\)

For the problems treated in this investigation, the above differential equations cannot be solved in a closed form so an approximate method of solution must be employed. This approximate solution was preferred to be the method of finite differences because of its simplicity,

\(^9\text{Ibid.}, \text{p. 461.}\)
ability to handle general boundary conditions and adaptability to
electronic computation.

Plate Stress Distribution

As the finite difference method was employed in this investi­
gation, the first step was to replace the surface of the plate by a
mesh or lattice system of individual nodal points. The next steps were
to calculate the stress function boundary values and the assignment
of an initial stress function approximate value for each nodal point
within the boundary.

The finite difference method essentially approximates partial
differential equations by linear algebraic equations in the form of
"operator molecules."* Where a mesh system of individual square
elements is being considered, the operator molecule for the biharmonic
equation (1) is:

\[ \nabla^4 \phi = \frac{1}{h^4} = 0 \quad (1a) \]

*See Appendix G.
where "h" is the mesh spacing. 10

When applied at each nodal point of the mesh, the biharmonic operator molecule yields a system of linear algebraic equations in $\phi$ which replaces the continuous biharmonic equation (1). Solving these equations simultaneously yields the approximate value of $\phi$ at each of the nodal points.

Knowing the final value of the stress function at each point of the mesh system, the stresses can be determined. The expressions for the stress components and their finite difference operator molecules are as shown below. 11

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{h^2}$$

(2a)

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = \frac{\nu}{h^2}$$

(3a)

10Ibid., p. 489.

The above stress molecules are employed by first locating each molecule central value over each nodal point. The next step is to sum algebraically the terms resulting from the multiplication of each molecule by its corresponding stress function value. In this manner each stress molecule yields a linear algebraic equation for the stress value.

After the stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ have been computed, the principal stresses and their directions can also be computed. The equations for the major and minor principal stresses and their directions are\(^ {12}\)

\[
\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (5)
\]

\[
\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (6)
\]

and

\[
\tan 2\theta = \frac{-2 \tau_{xy}}{\sigma_x - \sigma_y} \quad (7)
\]

where

\( \sigma_1 \) = the major principal stress

\( \sigma_2 \) = the minor principal stress

\( \theta \) = the angular orientation of the principal stress axes with respect to the X-axis.

All of the pertinent finite difference operator molecules and principal stress equations were computed electronically, using the electronic computer available at South Dakota State University.

There is now sufficient information available so that the first elastic stress distribution problem can be illustrated.
CHAPTER III

THE FULL PLATE COMPRESSION PROBLEM

PLATE A

The following analysis applies predominantly to plates but there is some application for sections of beams.

The General Boundary Stress Function Equations

Consider a rectangularly notched structural member subjected to a condition of axial compression as shown in Figure 1.

![Figure 1. The Full Plate Compression Problem](image)

Preliminary investigations have shown that such problems should be solved by considering one-half the plate and calculating the stress along the centerline of the member as shown in Figure 2.

The stress distribution at the plate centerline can be calculated from the expressions
Figure 2.
Boundary Stress Relationship
Plate A
\[-\sigma_r = - \frac{P}{A} - \frac{Mc}{I}\]  

and

\[\sigma_m = - \frac{P}{A} + \frac{Mc}{I}\]

where

\[P = 2R \sigma_n t\]
\[A = (T + V)t\]
\[M = 2R \sigma_n t(R - \frac{T + V}{2})\]
\[c = \frac{T + V}{2}\]

and

\[I = \frac{(T + V)^3}{12} t\]

where "t" is taken as the plate thickness. This problem is concerned with a tension-compression stress effect at the plate centerline. The location of the neutral axis can be determined from the relationship

\[T = (T + V) \frac{-\sigma_r}{\sigma_r - \sigma_m}\]  

(10)

Knowing the proper stress relationships, the general boundary stress function equations can be written.

Along the boundary B-H there acts the uniformly distributed compressive stress of intensity \(\sigma_n\). Therefore,

\[\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = - \sigma_n\]  

(11)
Integrating equation (11) twice with respect to $y$,

$$\frac{\partial \phi}{\partial y} = -n y + C_1 \tag{12}$$

and

$$\phi = -\frac{n y^2}{2} + C_1 y + C_2 \tag{13}$$

At point $A$ ($y = 0$),

$$\frac{\partial \phi}{\partial y} = C_1$$

and

$$\phi = C_2$$

The constants of integration $C_1$ and $C_2$ can be taken arbitrarily as zero because point $A$ is merely a starting or reference point for $\phi$ and $\frac{\partial \phi}{\partial y}$.

As the boundary surface $B-H$ is subject to a pure compressive stress in the $X$ direction, and no stresses in the $Y$ direction,

$$\frac{\partial^2 \phi}{\partial x^2} = 0$$

Integrating twice with respect to $x$,

$$\frac{\partial \phi}{\partial x} = C_3$$

and

$$\phi = C_3 x + C_4$$

However, along the surface $B-H$ ($x = 0$),

$$\frac{\partial \phi}{\partial x} = 0$$
so

$$C_3 = 0$$

Then, at point A ($x = 0, \ \phi = 0$),

$$C_4 = 0$$

Along the surface B-H shear stresses vanish, so

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

It can also be shown that the constants of integration are equal to zero.

Along the boundary surface B-H, as all constants of integration have been reduced to zero, equations (12) and (13) become,

$$\frac{\partial \phi}{\partial y} = - \alpha_n y$$  \hspace{1cm} (14)

and

$$\phi = - \frac{\alpha_n y^2}{2}$$  \hspace{1cm} (15)

where $\alpha_n$ can be any convenient constant value.

The stress function values can be determined at points B and H by substituting into equations (14) and (15).

At point B,

$$\frac{\partial \phi}{\partial y} = - \alpha_n R$$  \hspace{1cm} (14a)

and

$$\phi = - \frac{\alpha_n R^2}{2}$$  \hspace{1cm} (15a)
At point H,

\[ \frac{\partial \phi}{\partial y} = - \infty n R \]  

(14b)

and

\[ \phi = - \frac{\infty n R^2}{2} \]  

(15b)

Equation (14) shows that the slope of the stress function varies linearly with the distance from the neutral axis. Equation (15) indicates that the stress function is a second degree parabola.

There are no normal stresses applied along the boundary H-G. Therefore,

\[ \frac{\partial^2 \phi}{\partial x^2} = 0 \]

Integrating twice with respect to x,

\[ \frac{\partial \phi}{\partial x} = c_5 \]  

(16)

and

\[ \phi = c_5 x + c_6 \]  

(17)

But, at point H (x = 0),

\[ \frac{\partial \phi}{\partial x} = 0 \]  

(16a)

so

\[ c_5 = 0 \]
For the boundary $H-G$, equation (17) simplifies to the expression,

$$\phi = C_0$$

$$= - \frac{\alpha \pi R^2}{2}$$

= the value of $\phi$ at point $H$ from equation (15b).

Then, at point $G$,

$$\phi = - \frac{\alpha \pi R^2}{2}$$

$$= \frac{\partial \phi}{\partial x} = 0$$

(16a)

$$= \frac{\partial \phi}{\partial y} = - \alpha \pi R$$

(19)

= the same value as for point $H$ from equation (14b).

Similarly, for the line $B-C$,

$$\frac{\partial \phi}{\partial x} = 0$$

(16a)

and

$$\phi = \text{a constant} = - \frac{\alpha \pi R^2}{2}$$

Then at point $C$,

$$= - \frac{\alpha \pi R^2}{2}$$

(20)

= the same value for $\phi$ as at point $B$ from equation (15a).
Also

\[ \frac{\partial \phi}{\partial x} = 0 \]

and

\[ \frac{\partial \phi}{\partial y} = -\alpha_n R \]  \hspace{1cm} (19b)

= the same value as for point B from equation (14a).

The above analysis indicates that the values of the stress function for surfaces H-G and B-C are constant and equal to the stress function values calculated for points H and B respectively.

Along the surface G-F, as there are no stresses applied,

\[ \frac{\partial^2 \phi}{\partial y^2} = 0 \]

Integrating twice with respect to y,

\[ \frac{\partial \phi}{\partial y} = C_7 \]  \hspace{1cm} (19c)

and

\[ \phi = C_7 y + C_8 \]  \hspace{1cm} (21)

But, at point G,

\[ \frac{\partial \phi}{\partial y} = -\alpha_n R \]  \hspace{1cm} (19d)

so

\[ C_7 = -\alpha_n R \]

Equation (21) simplifies to the expression

\[ = -\alpha_n R y + C_8 \]
But, for point G,

\[ \sigma_n R^2 = - \frac{\sigma_n R^2}{2} \]  

(18)

so

\[ - \frac{\sigma_n R^2}{2} = - \sigma_n R^2 + C_8 \]

and

\[ C_8 = \frac{\sigma_n R^2}{2} \]

Therefore, equation (21) becomes, for the boundary G-F,

\[ \varphi = - \sigma_n R y + \frac{\sigma_n R^2}{2} \]  

(22)

Then, at point F,

\[ \frac{\partial \varphi}{\partial y} = - \sigma_n R \]  

(19e)

and

\[ \varphi = - \sigma_n R w + \frac{\sigma_n R^2}{2} \]  

(23)

There are no stresses applied along the boundary F-E. Therefore,

\[ \frac{\partial^2 \varphi}{\partial x^2} = 0 \]

Integrating twice with respect to x,

\[ \frac{\partial \varphi}{\partial x} = C_9 \]

and

\[ \varphi = C_9 x + C_{10} \]

However, at point F, from a continuation of equation (16a) along the boundary G-F,
\[ \frac{\partial \phi}{\partial x} = 0 \]

so

\[ c_9 = 0 \]

and

\[ \phi = c_{10} \]

\[ = - \alpha n R W + \frac{\alpha n^2}{2} \quad (24) \]

= the same value as for point F from equation (23).

So, for point E,

\[ \frac{\partial \phi}{\partial x} = 0 \]

\[ \frac{\partial \phi}{\partial y} = - \alpha n R \quad (19f) \]

= the same value as for point F from equation (19e).

Then,

\[ \phi = - \alpha n R W + \frac{\alpha n^2}{2} \quad (24a) \]

The above completes the boundary stress function equations for one-half the plate. The stress function boundary values for the left half of the plate will be identical to those for the right half because of symmetry.

For this problem, where a tension-compression effect exists along the centerline of the plate, \( \alpha_x \) equals zero at some point D. Then, along the surface D-C,

\[ \alpha_x = \frac{\partial^2 \phi}{\partial s^2} = \frac{\alpha m m_n}{s} \quad (25) \]
Integrating twice with respect to \( s \),
\[
\frac{\partial \phi}{\partial s} = \frac{m \cdot n \cdot s^2}{2V} + C_{11}
\]
(26)
and
\[
\phi = \frac{m \cdot n \cdot s^3}{6V} + C_{11}s + C_{12}
\]
(27)
where the constants of integration, \( C_{11} \) and \( C_{12} \), can be evaluated from the known values of \( \frac{\partial \phi}{\partial y} \) and \( \phi \), respectively, at point \( C \).

For the surface \( D-E \),
\[
\phi_x = \frac{\partial^2 \phi}{\partial s^2} = -\frac{m \cdot n \cdot s}{T}
\]
(28)
Integrating twice with respect to \( s \),
\[
\frac{\partial \phi}{\partial s} = -\frac{m \cdot n \cdot s^2}{2T} + C_{13}
\]
(29)
and
\[
= -\frac{m \cdot n \cdot s^3}{6T} + C_{13}s + C_{14}
\]
(30)
where the constants of integration, \( C_{13} \) and \( C_{14} \), can be evaluated from the known value of \( \frac{\partial \phi}{\partial y} \) at point \( E \) and the known value of \( \phi \) at point \( D \), respectively.

As this analysis deals with the full plate section, equations (27) and (30) should be employed only to calculate values of the stress function for the points immediately outside the boundary points \( C \) and \( E \).
Stress Function Values Outside the Boundaries

The biharmonic molecule must be applied to the interior points adjacent to the boundary. Consequently, values of the stress function must be extrapolated for points immediately outside the boundaries. These points, outside the boundaries, are also employed for computing \( \tau_x, \tau_y \) and \( \tau_{xy} \) when the stress molecules are placed on the boundary points.

The necessary values for the stress function outside the boundary B-H can be taken identical to the values on the boundary. This assumption is valid because the member can be thought of as a continuous plate or a section of a continuous beam.

For the line B-C, the value beyond point B can be determined by extrapolation of the second degree parabola along A-B and the value of the stress function beyond point C can be obtained by extrapolation of the cubic parabola along D-C. This method has proved to be a more realistic approach than keeping the stress function values, beyond the boundary B-C, equal to the value determined immediately outside point B.

For the boundary H-G, the stress function value beyond point H can be determined by extrapolation of the second degree parabola along A-H. The stress function value beyond point G can be obtained from the extrapolation of the linear function along the boundary F-G.

For the surface G-F, the stress function values outside the boundary can be taken equal to those on the boundary.
The stress function value immediately beyond point $E$ can be extrapolated from the cubic function along $D-E$.

The values for points immediately beyond boundaries $B-C$, $H-G$ and $F-E$ can be determined on the assumption that the values of the stress function between each two end points vary linearly. It is realized that this assumption is not consistent with the assumed stresses on the boundaries. The rate of change of warping, along the above boundaries, varies slightly due to the existence of the notch. The inconsistency involved in a linear assumption does not have an appreciable effect.

The Computation Method

The general equations and conditions for calculating the stress function values for the boundaries, and the points immediately beyond the boundaries, are now available. The general boundary equations have been developed for a combination tension-compression effect at the plate centerline. Specific problems can be solved, using the general equations, by introducing constants for the variable plate dimensions.

In order to illustrate the full plate compression problem, a particular example (Plate $A$) problem was worked. The computed values are shown in Appendix $A$. After the equations for the boundary stress function values were obtained, the surface of the plate was replaced by a mesh or lattice system. The stress function values were calculated for each discrete point on the boundaries. The stress function values
immediately outside the boundaries were then determined. The next step was to assign an initial stress function value for each nodal point within the boundaries. At this stage each nodal point had a stress function value and this information was taken as computer input.

The full plate computer programs and flow diagrams are shown in Appendix B. The complete program was broken into pass one and pass two because the complete program, in one part, exceeded the 40,000 available units of core storage.

An examination of the pass one program, or the flow diagram, will show that the program employs an iterative technique to satisfy the biharmonic operator molecule. The biharmonic molecule was continually swept over the surface of the plate. The existing stress function value, at each nodal point, was replaced on each sweep by the computed value. Using this iterative process, the stress function values inside the plate boundaries were "forced" to converge to suitable values dictated by the stress function values on and immediately outside the boundaries, which do not change. As the iteration process converges, the biharmonic molecule becomes more nearly satisfied at each nodal point within the domain. That is, the residual for each nodal point iterated approaches zero.

In Plate A (example given in Appendix A) there are 279 nodal points within the domain. The machine time required for one iteration cycle over the entire plate was 85 seconds. The total amount of computer time which was necessary to approximately satisfy the biharmonic
function was seven hours. The maximum difference between any two successive iterations gave a maximum error of 0.047, which was considered satisfactory. If the computations were done by hand or even with a desk calculator, this accuracy would have never been obtained except after several months of continuous work.
CHAPTER IV

THE FULL PLATE MOMENT PROBLEM

PLATE B

The member treated in this section, Plate B, has the same general outlines as Plate A. The only change is the condition introduced at the boundaries.

The General Boundary Stress Function Equations

The rectangularly notched structural member, shown in Figure 3, is subject to a stress condition which approximates a condition of pure moment.

Figure 3. The Full Plate Moment Problem

Similar to Plate A, it was preferred to consider one-half the plate where the stress along the centerline was calculated as shown in Figure 4. This stress can be calculated from the fact that the moment
Figure 4.
Boundary Stress Relationship
Plate B
introduced on the section boundaries causes an equal moment at the centerline of the section.

Then,

$$2\left(\frac{\alpha_m}{2}\right)\left(\frac{R}{2}\right)\frac{T}{2}(t) = 2\left(\frac{\alpha_n}{2}\right)\left(\frac{R}{2}\right)\frac{T}{2}(t) \quad (31)$$

and

$$\alpha_m = \pm \frac{R^2}{T^2} \alpha_n \quad (32)$$

where "t" is the plate thickness. Knowing the proper stress relationships, the general boundary stress function equations can be written.

Along the boundary B-H there acts a triangular stress distribution where

$$\alpha_x = \frac{\partial^2 \phi}{\partial y^2} = \pm \frac{\alpha_n y}{R} \quad (33)$$

The sign depends on whether the boundary condition is positive or negative. Integrating equation (33) twice with respect to y,

$$\frac{\partial \phi}{\partial y} = \pm \frac{\alpha_n y^2}{2R} + C_1 \quad (34)$$

and

$$\phi = \pm \frac{\alpha_n y^3}{6R} + C_1 y + C_2 \quad (35)$$

Then, at point A ($y = 0$),

$$\frac{\partial \phi}{\partial y} = C_1$$

and

$$\phi = C_2$$
The constants of integration $C_1$ and $C_2$ can be taken arbitrarily as zero because point A is a reference point for $\phi$ and $\frac{\partial \phi}{\partial y}$.

As the boundary surface B-H is subject to normal stresses only, it can be taken as a free surface.

Then,

$$\frac{\partial^2 \phi}{\partial x^2} = 0$$

and

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Employing a method of analysis similar to that used for Plate A, the constants of integration, for the above two equations, can be taken as zero.

For the boundary surface B-H, as all constants of integration have been reduced to zero, equations (34) and (35) are reduced to,

$$\frac{\partial \phi}{\partial y} = \pm \frac{\alpha ny^2}{2R}$$

(36)

and

$$\phi = \pm \frac{\alpha ny^3}{6R}$$

(37)

It is more convenient to assign two constant values, $K_1$ and $K_2$, for the constant quantities in equations (37) and (36), respectively. Then,

$$\frac{\alpha n}{6R} = K_1$$

(38)
and

\[ \frac{\partial^2 \phi}{\partial y^2} = \pm K_2 y^2 \]  

(40)

and

\[ \phi = \pm K_1 y^3 \]  

(41)

The signs in equations (40) and (41) depend upon the boundary condition. Equation (40) indicates that the slope of the stress function is a second degree parabola and equation (41) shows that the stress function is a third degree parabola.

The stress function and slope can be obtained at points B and H by substituting into equations (40) and (41).

At point B,

\[ \frac{\partial^2 \phi}{\partial y^2} = K_2 R^2 \]  

(40a)

and

\[ \phi = K_1 R^3 \]  

(41a)

At point H,

\[ \frac{\partial^2 \phi}{\partial y^2} = -K_2 R^2 \]  

(40b)

and

\[ \phi = -K_1 R^3 \]  

(41b)
For the surface B-C, there are no boundary conditions applied. Then, following a procedure similar to that used for Plate A, the values of $\frac{\partial \phi}{\partial y}$ and $\phi$ are constant along B-C and identical to the values obtained at point B.

Then, at point C,

$$\frac{\partial \phi}{\partial y} = K_2 R^2 \quad (40a)$$

and

$$\phi = K_1 R^3 \quad (41a)$$

The values of $\frac{\partial \phi}{\partial y}$ and $\phi$ are constant along H-G and identical to the values at point H.

Then, at point G,

$$\frac{\partial \phi}{\partial y} = -K_2 R^2 \quad (40c)$$

and

$$\phi = -K_1 R^3 \quad (41b)$$

There are no stresses applied along the surface G-F. Then,

$$\frac{\partial^2 \phi}{\partial y^2} = 0$$

Integrating twice with respect to $y$,

$$\frac{\partial \phi}{\partial y} = C_3 \quad (42)$$

and

$$\phi = C_3 y + C_4 \quad (43)$$
But, at point G,
\[
\frac{\partial \phi}{\partial y} = - K_2 R^2 \tag{40c}
\]
so
\[
C_3 = - K_2 R^2
\]
Equation (43) simplifies to the expression
\[
\phi = - K_2 R^2 y + C_4
\]
But, for point G,
\[
\phi = - K_1 R^3 \tag{41b}
\]
so
\[
- K_1 R^3 = - K_2 R^3 + C_4
\]
and
\[
C_4 = R^3(K_2 - K_1)
\]
Therefore, equation (43) becomes, for the surface G-F,
\[
\phi = - K_2 R^2 y + R^3(K_2 - K_1) \tag{43a}
\]
Then, at point F,
\[
\frac{\partial \phi}{\partial y} = - K_2 R^2 \tag{40d}
\]
and
\[
\phi = - K_2 R^2 W + R^3(K_2 - K_1) \tag{44}
\]
There are no stresses applied to the boundary F-E. Employing a procedure similar to that used for Plate A, the values of \(\frac{\partial \phi}{\partial y}\) and \(\phi\) remain constant at the values obtained for point F.
Then, at point E,
\[
\frac{\partial \phi}{\partial y} = - K_2 R^2 \tag{40e}
\]
and

$$\phi = -K_2R^2w + \frac{3}{2}(K_2 - K_1) \quad \text{(44a)}$$

The above completes the boundary stress function equations for one-half the plate. The stress function boundary values for the left half of the plate are identical to those for the right half because of symmetry.

For this problem, where a tension-compression effect exists along the centerline of the plate, \(\phi_x\) equals zero at some point D. Continuing at point D, it is noted that a triangular stress distribution exists where

$$\phi_x = \frac{\partial^2 \phi}{\partial s^2} = \pm \frac{m s}{T} \quad \text{(45)}$$

The sign is governed by whether the boundary condition is positive or negative. Integrating equation (45) twice with respect to \(s\),

$$\frac{\partial \phi}{\partial s} = \pm \frac{m s^2}{2T} + C_5 \quad \text{(46)}$$

and

$$\phi = \pm \frac{m s^3}{6T} + C_5s + C_6 \quad \text{(47)}$$

But, for the surfaces B-H,

$$\frac{\phi_n}{6R} = K_1 \quad \text{(38)}$$

and

$$\frac{\phi_n}{2R} = K_2 \quad \text{(39)}$$
Therefore, as \( \alpha_n \) can be assigned any convenient value,

\[
K_1 = \frac{\alpha_n}{6R} = \frac{\alpha_n}{K_3}
\]  

and

\[
K_2 = \frac{\alpha_n}{2R} = \frac{\alpha_n}{K_4}
\]

Similarly, along side G-E,

\[
\frac{\alpha_m}{6T} = \frac{\alpha_m}{K_5}
\]

and

\[
\frac{\alpha_m}{2T} = \frac{\alpha_m}{K_6}
\]

As the constant \( K_1 \), for boundary B-H, may be assigned any convenient value and as \( \alpha_n \) can be normalized to unity,

\[
\frac{\alpha_m}{6T} = \frac{K_1}{K_5}
\]

so

\[
\frac{\alpha_m}{6T} = \frac{K_1 K_3 \alpha_m}{K_5}
\]

Similarly,

\[
\frac{\alpha_m}{2T} = \frac{K_2}{K_6}
\]
Therefore, along the surface C-E, employing equations (46) and (47),

\[
\frac{\partial \phi}{\partial s} = \frac{K_{2}K_{4}}{2T} \frac{m}{K_{6}} s^{2} + C_{5} \tag{46a}
\]

and

\[
\phi = \frac{K_{1}K_{5}}{2K_{6}} \frac{m}{K_{6}} s^{3} + C_{5}s + C_{6} \tag{47a}
\]

In regard to the surface D-C, the constants of integration, \(C_{5}\) and \(C_{6}\), can be evaluated from the known values of \(\frac{\partial \phi}{\partial y}\) and \(\phi\), respectively, at point C. Concerning surface D-E, the constants \(C_{5}\) and \(C_{6}\) can be evaluated from the known value of \(\frac{\partial \phi}{\partial y}\) at point E and the known value of \(\phi\) at point D, respectively. As this analysis deals with the full plate section, equations (46a) and (47a) should be employed only to calculate values of the stress function at the points immediately outside the boundary points C and E.

The stress function boundary values can be calculated for a particular example problem by using the equations given above. The stress function values outside the boundaries can be determined by linear extrapolation between the points beyond the boundary. This manner of calculating the stress function values outside the boundaries is nearly identical to that employed for Plate A.
In order to illustrate the full plate moment problem, an example problem (Plate B) was computed using the full plate computer programs listed in Appendix B. The computed results are shown in Appendix A.
CHAPTER V

THE HALF PLATE PROBLEMS

Accuracy Considerations

Up to this point, the two problems computed are the full plate compression and moment problems. These computed problems are shown as Plate A and Plate B, respectively, in Appendix A. A study of the above two numerical examples seems to indicate that although the results are logical and helpful, the accuracy should be improved. The most practical method of obtaining more accurate results, in regard to the finite difference method, is to decrease the distance between the lattice points. It can be shown that the error of the derivative of the stress function, due to the employment of finite differences rather than differential equations, is proportional to the square of the mesh side, when this is small.\(^{13}\)

In an effort to employ the above information in order to improve on the degree of accuracy obtained from the full plate problems, it was decided to work "the half plate problems." These half plate problems consider the identical sections and loading conditions as the full plate problems but deal with one-half the symmetrical member and employ a reduced mesh size.

The most logical method of decreasing the mesh size, for the conditions of this investigation, is to reduce it by one-half. This method of mesh size reduction was used here. This procedure will render a check to corresponding points on both the full and half plate problems. This ability to check the full section versus the half section can give rise to various conclusions as to whether it is best to work with either section as well as some indication of how many points, or what mesh size is desirable in order to obtain reasonable accuracy for problems similar to those investigated in this work.

The mesh size was not halved on the full plate sections because of increased computer core storage problems. However, this method of approach would be highly desirable if core storage facilities permit.

The Half Plate Compression Problem (Plate C)

In an attempt to improve on the results of the full plate compression problem and in accordance with the accuracy considerations set forth above, this section is concerned with the half plate compression problem. In order to illustrate the method of calculating specific boundary stress function values, Appendix A contains the numerical calculations for Plate C. The computed stresses are also shown in Appendix A.

Of major interest are the stress function values, the maximum principal stresses and their directions. Contours for the stress function and the maximum principal stresses have been drawn and are shown within Figures 5 and 6, respectively. A graphical representation
Figure 5.
Stress Function
Plate C
Figure 6.
Maximum Principal Stresses
Plate C
Figure 7.
Principal Stress Orientation
Plate C
of the orientation of the principal stress planes is shown within Figure 7.

The Half Plate Moment Problem (Plate D)

In an attempt to improve on the results of the full plate moment problem and in accordance with the accuracy considerations set forth above, this section is concerned with the half plate moment problem. In order to illustrate the half plate moment problem, Appendix A includes the method of calculating specific boundary stress function values. The computed stresses are also shown in Appendix A.

Information of primary interest, in the form of contours for the final stress function values and the maximum principal stresses, are shown within Figures 8 and 9, respectively. A graphical representation of the principal stress plane orientations is shown within Figure 10.

The Half Plate Computer Programs

The half plate computer programs are shown in Appendix B. An examination of the pass one program shows that it is similar to pass one of the full plate programs. However, the half plate program has a sense switch statement which permits the convergence difference and decimal difference to be printed or not at the operators choice.

For the half plate example problems, the machine time required for completion of the iterative procedure was approximately fifteen hours. The electronic computer required a time period of 175 seconds
Figure 8.
Stress Function
Plate D
Figure 9.
Maximum Principal Stresses
Plate D
Figure 10.
Principal Stress Orientation
Plate D
to perform one complete plate cycle of the iteration process. This amount of time was required to satisfy the biharmonic equation to a maximum difference between any two successive iterations of 0.23. At this point, the maximum difference between any two successive iterations was approximately four percent. With this degree of accuracy the progression of the iteration convergence was nearly negligible. An examination of the results indicated that the accuracy was sufficient.

As was the case for the previous full plate example problems, after the pass one program had been completed, the results were taken as input to the pass two program. Pass two of the half plate programs is very similar to pass two of the full plate programs except that the half plate program employs a partitioned mesh system.

After both passes of the half plate programs had been completed, the punched card output was taken as input to an accounting machine in order for the computed results to be shown on the plan of the plate.
CHAPTER VI

PHOTOELASTICITY

In an attempt to substantiate some aspects of the analytical stress distribution, a photoelastic beam-model of the structural member was constructed and tested. The model was constructed from a special photoelastic material known as Catalina-Resin (CR-39).

As concerns the model construction, the first step was to machine an aluminum template to the desired scale. The model material was then taped to the template using double-sided tape. The model was machined using a very high speed router which was commercially designed specifically for the machining of photoelastic materials. This method of machining provided very smooth, flat edge surfaces on which time-edge effects were almost non-existent.

The beam-model was subjected to a condition of pure moment, with the notch area in compression. The resulting photographs are shown in Figure 11.

The first region, in pure bending, to show photoelastic effects was the notch upper corner. The next regions to show stress effects were the horizontal fibers at the notch centerline.

Inspection of the stress pattern photographs show several interesting characteristics. It should be noted that the neutral surface is not continuous throughout the length of the model but that it terminates in the vicinity of the notch upper corners. This effect was
Figure 11. Pure Moment Photoelasticity Photographs
also noted by Frocht\textsuperscript{14} when testing a model and loading similar to those investigated here. It should also be noted that the fringe orders at the notch top corners are higher than on the straight central portions and the notch bottom corners show a condition of zero stress. It is also evident that the central portion shows fringe orders that are nearly horizontal, parallel and equidistant which indicates a condition of pure bending. However, the stress patterns outside the notch vicinity show the presence of a combined stress condition. The above condition is also evident from photographs of a similar model by Frocht.\textsuperscript{15}

For this study it was not attempted to evaluate the fringe order or the stress concentration values. The above is due to the following reasons:

(1) The main objective of the photoelastic study was to obtain an approximate check on the distribution of the analytical stress contours.

(2) There is some disagreement within the scientific field as to whether the fringes actually represent stress contours, strain contours or contours of maximum shear.

(3) The termination of the neutral surface, in the vicinity of the notch corner, may be due to the localized buckling


\textsuperscript{15} Ibid., p. 143.
and plastic flow which takes place at such a sharp corner. If the stress relationship in this area is not nearly linear, the fringe values and stress concentration factors would be of a very doubtful degree of accuracy.
CHAPTER VII

GENERAL SUMMARY

Discussion

The primary purpose of this work was to investigate the elastic stress distribution in rectangularly notched structural members. The analysis employed the stress function, the finite difference method and electronic computation. The structural members were investigated for the common loading conditions of axial compression and pure moment. Some photoelastic work was attempted in order to verify the analysis.

This basic research work can be thought of as a base for future investigations. Similar problems can be solved by using or modifying the basic methods and the computer programs.

The stress distribution analysis was based upon an assumed stress distribution along the boundaries of the plate. The plate was then reduced to a set of discrete points and the stress function values were calculated at each point on the boundaries. Other stress function values, for points within the domain, were determined by satisfying the biharmonic equation at each point. These interior stress function values depend upon the stress function values on the boundaries, the values beyond the boundaries, and each other. The plate stresses were then determined.

The finite difference method indicates that the plate sections, to the sides of the notch, act as cantilevers. Once the cantilever
action was recognized to be present, the moment and shear conditions were checked against the boundary conditions applied. Equilibrium was satisfied within ten percent, assuming a pure cantilever condition. This is not surprising because of the continuity existing from a continuous plate.

The half plate examples show a stress concentration factor of 6.44 for compression and 3.89 for moment. The first value is in fair agreement with Seely and Smith\(^\text{16}\) who give an experimental stress concentration factor of approximately 7.00 for a triangularly notched member. The triangularly shaped notch had sharp corners and the member was subject to a uniformly distributed tension condition. However, there is no mathematical solution for the theoretical stress concentration factor at a sharp notch.\(^\text{17}\)

A check was made on the full plate versus the half plate problems. The identical loading conditions check each other within eight percent, on the average, for the various quantities computed.

It has been stated earlier\(^*\) that the finite difference method seems to have an accuracy of approximately two percent when compared to the more exact stress function solution. This degree of accuracy


\(^{17}\) Ibid., p. 395.

\(^*\)See previous investigations section.
was obtained with mesh systems as large as 24 x 24. However, this degree of accuracy is probably only obtainable for sections which have a simple rectangular outline shape.

Conclusions

The following conclusions may be drawn.

(1) The principal stress contours indicate that the notch corner is a point of high stress concentration.

(2) It is apparent that the cantilever action appeared without any provision being made for it when calculating the boundary stress function values. This indicates that the stress function method of analysis is a powerful tool for determination of structural behavior, without the need of special conditions or provisions.

(3) A mesh system size of 20 x 20 appears to be about the ideal size for use with the Fortran II computer language and a core storage of approximately 40,000.

(4) The computer time, required for the iteration process, can be reduced considerably by resorting to a lesser degree of accuracy, which would be sufficient for ordinary engineering design.

(5) A rectangular notch has a point of singularity, at the notch corner, and the stress values at the corner cannot be determined. This condition has some influence on the accuracy of the computed stresses in the notch vicinity.
Nevertheless, for points of some small distance from the notch, the degree of accuracy is thought to be about eight percent or better. For areas of some distance from the notch, the degree of accuracy probably approaches two percent.

(6) The illustrative samples given in Appendix A show that the half plate problems are more accurate than the full plate problems. This is not surprising because the smaller the mesh size the better the accuracy.

(7) The finite difference method of stress distribution analysis has produced relatively accurate data, considering the complexities of the problems treated. There are other errors inherited from the finite difference approximations and other types of errors such as the degree of iterative convergence.

Future Areas of Study and Research

This paper presents the results of a pilot investigation of two notched structural members with rectangular cross sections and rectangular notches subject to conditions of pure moment and axial compression. The use of the stress function, coupled with a numerical technique and electronic computation, is relatively new. It is felt that there are many more problems still in need of further detailed study and many other problems which have not yet been attempted, for
which a similar approach can be employed. Some of the future areas of study and research are listed below.

(1) This study was restricted to rectangular structural sections only. Additional studies could include rolled sections of various shapes.

(2) It has been shown that the stress function values immediately outside the boundaries are of great importance. However, the published information for their accurate extrapolation is somewhat lacking.

(3) The biharmonic iteration process is somewhat lengthy. A possible improvement is to program the biharmonic solution, or the entire solution, in the Symbolic Programming System (SPS) language. The SPS language may also give some relief from core storage difficulties as were encountered in this work. However, the use of the SPS language may not be all gain. There may be data input and output problems as well as resulting programs which may be less flexible. Possibly it is better to use a faster computer with a larger storage capacity.

(4) This investigation was restricted to a study of the stress distribution in notched structural members. Knowing the stress distributions, the problems should be extended to the determination of the critical buckling characteristics.
The buckling characteristics may be especially critical for light gage or high strength steel members.

(5) The photoelastic analysis indicated that the neutral surface terminates in the vicinity of the notch corner. Some future work is needed to determine why this condition is evident.
BIBLIOGRAPHY


APPENDIX A

EXAMPLE BOUNDARY STRESS FUNCTION CALCULATIONS

AND

COMPUTED STRESS DISTRIBUTION DATA
EXAMPLE ONE

THE FULL PLATE COMPRESSION PROBLEM

PLATE A

COMPUTED STRESS DISTRIBUTION DATA
Figure 12. Stress Function, $\phi$, For Pure Compression

Plate A

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Figure 14. Stress, $\sigma_y$, For Pure Compression

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**Figure 15.** Shearing Stress, $\tau_{xy}$, For Pure Compression Plate A
| B | 08 | 137 | 72 | 33 | 1.23 | 95 | 2.20 | 2.87 | 3.12 | 3.93 | 6.20 | 14.86 | 27.08 | 40.26 | 58.47 | 95 | 120 | 37 | 57 | 141 | 10 | 73 |
|---|----|-----|----|----|------|-----|-----|------|------|------|------|--------|--------|--------|------|-----|----|----|-------|----|-----|
| 2.67 | 8.16 | 14.13 | 20.90 | 28.77 | 38.19 | 40.17 | -26.92 | -14.61 | -5.93 | 8.52 | 2.56 | 1.18 | -1.69 | -4.96 | 0.00 | -1.09 | 6.20 | -8.69 | -2.56 | -15 | 1.72 | 4.97 | 0.00 | 0.00 | 0.00 |
| 1.96 | 5.99 | 13.51 | 16.36 | 24.17 | 27.08 | 17.21 | 9.11 | 4.96 | 2.56 | 8.52 | 2.56 | 1.18 | -1.69 | -4.96 | 0.00 | 0.01 | 1.3 | -0.08 | 0.00 | -0.01 | 0.01 | 0.00 | 0.00 | 0.00 |
| 3.60 | 2.20 | 24.08 | 10.00 | 11.73 | 20.04 | 17.21 | -14 | 2.20 | 2.67 | 14.86 | 27.08 | -9.15 | 16 | 6.76 | 8.16 | -9.15 | 27.08 | -27.68 | -17.27 | -7.54 | 4.48 | 14.05 | -19.31 | 43.54 | 1.06 | 45.00 |
| 2.20 | 1.77 | -13.86 | -17.12 | -14.92 | -11.10 | -38.46 | -44.06 | -41.20 | 37.33 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

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Figure 17. Maximum Principal Stress, \( \sigma_p \), For Pure Compression
Plate A
EXAMPLE TWO

THE FULL PLATE MOMENT PROBLEM

PLATE B

COMPUTED STRESS DISTRIBUTION DATA
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Figure 19. Stress, $\sigma_x$, for Pure Bending
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Figure 21. Shearing Stress, $\tau_{xy}$, For Pure Bending
Plate B
<table>
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<th>13.76</th>
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<td>.02</td>
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<td>.04</td>
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<tr>
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<td>5.13</td>
<td>4.92</td>
<td>2.59</td>
<td>78</td>
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</tr>
</tbody>
</table>

Figure 22. Principal Stress Orientation, Θ, For Pure Bending Plate B
Figure 23. Maximum Principal Stress, $\sigma_p$, For Pure Bending
Plate B
EXAMPLE THREE

THE HALF PLATE COMPRESSION PROBLEM

PLATE C

EXAMPLE BOUNDARY STRESS FUNCTION CALCULATIONS

AND

COMPUTED STRESS DISTRIBUTION DATA
EXAMPLE BOUNDARY STRESS FUNCTION CALCULATIONS

Consider the half plate section, dimensioned in terms of the mesh spacing, "h," as shown in Figure 24. The mesh system employed is shown in Figure 25, where the plate boundaries are shown as the heavy solid lines.

Plate C is subject, as was Plate A, to a pure compressive stress. Plate C has the same general boundary outlines as Plate A. The basic boundary equations, determined for Plate A, will be employed here as needed.

Substituting into equation (8):

\[-\sigma_r = - 5 \sigma_n\]

Substituting into equation (9):

\[\sigma_m = + \frac{5}{3} \sigma_n\]

Substituting into equation (10):

\[T = 9\]

so

\[V = 3\]

Employing equation (15) to calculate stress function values along the boundary B-H, where \(\sigma_n\) is taken equal to 60:
Figure 24.
Boundary Stress Relationship
Plate C
Figure 25.
Half Plate Mesh System
The value of \( \tau_n \) was taken to be 60 in order to facilitate the use of the generalized half plate computer programs shown in Appendix B. This procedure produces normalized stresses, with respect to \( \tau_x \), on the boundary B-H.

At point G, from equations (19) and (18),

\[
\frac{\partial \phi}{\partial y} = - \tau_n R = -600
\]

\[
= - \frac{\tau_n R^2}{2} = -3000
\]

and the stress function values remain constant at -3000 along the surface H-G.

At point C, from equations (19b) and (20),

\[
\frac{\partial \phi}{\partial y} = - \tau_n R = -600
\]

\[
\phi = - \frac{\tau_n R^2}{2} = -3000
\]
and the stress function values remain constant at -3000 along the surface B-C.

For the surface G-F, employing equation (22):

<table>
<thead>
<tr>
<th>Point</th>
<th>y_1</th>
<th>( \phi )</th>
<th>y_2</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>G'</td>
<td>11</td>
<td>-3600</td>
<td>6</td>
<td>-600</td>
</tr>
<tr>
<td>G</td>
<td>10</td>
<td>-3000</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-2400</td>
<td>4</td>
<td>600</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-1800</td>
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<tr>
<td></td>
<td>7</td>
<td>-1200</td>
<td>2</td>
<td>1800</td>
</tr>
</tbody>
</table>

At points F and E, from equations (19e) and (24),

\[
\frac{\partial \phi}{\partial y} = - \alpha_n R = -600
\]

\[
\phi = - \alpha_n R + \frac{\alpha_n R^2}{2} = 1800
\]

and the stress function values remain constant at 1800 along the surface F-E.

For the surface D-C,

\[
\frac{\partial \phi}{\partial s} = \frac{\alpha_m \alpha_n s^2}{2V} + C_{11}
\]  \(26\)

Also, for point C,

\[
\phi = -3000
\]

\[
\frac{\partial \phi}{\partial y} = -600
\]

\[
\alpha_m = \frac{5}{3}
\]
\( n = 60 \)

and

\[ s = V = 3 \]

Substitution of the above quantities, for point C, into equation (26):

\[ c_{11} = -750 \]

Also, for the surface D-C,

\[ \phi = \frac{m n s^2}{6V} + c_{11}s + c_{12} \]  

(27)

Substituting the above quantities, for point C, into equation (27):

\[ c_{12} = -900 \]

Therefore, along the surface D-C,

\[ \phi = \frac{50}{9} s^3 - 750 s - 900 \]  

(27a)

Substituting values of \( s \) into equation (27a), for surface D-C:

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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</tr>
<tr>
<td>3</td>
<td>-3000</td>
</tr>
<tr>
<td>2</td>
<td>-2356</td>
</tr>
<tr>
<td>1</td>
<td>-1644</td>
</tr>
<tr>
<td>0</td>
<td>-900</td>
</tr>
</tbody>
</table>

For the surface D-E,

\[ \frac{\partial \phi}{\partial s} = \frac{m n s^2}{2T} + c_{13} \]  

(29)
Also, for point E,
\[ \phi = 1800 \]
\[ \frac{\partial \phi}{\partial y} = -600 \]
\[ -\sigma_r = -5 \]
\[ \sigma_n = 60 \]

and
\[ s = T = 9 \]

Substitution of the above quantities, for point E, into equation (29):
\[ C_{13} = 750 \]

Also, for the surface D-E,
\[ \phi = - \frac{\sigma_r \sigma_n s^3}{6T} + C_{13} s + C_{14} \] (30)

Substituting the appropriate quantities, for point D, into equation (30):
\[ C_{14} = -900 \]

Therefore, along the surface D-E,
\[ \phi = - \frac{50}{9} s^3 + 750 s - 900 \] (30a)

Substituting values of s into equation (30a), for surface D-E:

<table>
<thead>
<tr>
<th>s</th>
<th>( \phi )</th>
<th>s</th>
<th>( \phi )</th>
</tr>
</thead>
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<tr>
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<td>6</td>
<td>2400</td>
</tr>
<tr>
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<td>-156</td>
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<td>2444</td>
</tr>
<tr>
<td>2</td>
<td>556</td>
<td>8</td>
<td>2255</td>
</tr>
<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>1744</td>
<td>10</td>
<td>1045 Pt. E'</td>
</tr>
<tr>
<td>5</td>
<td>2156</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The above completes the boundary stress function calculations.

With reference to Figure 25, the stress function values immediately beyond the boundaries B-H, F-G and C-E were extrapolated equal to those on the boundaries. The values immediately beyond the boundary B-C were taken as a linear relationship between the extrapolated values at points B' and C'. In a similar manner, the values immediately beyond boundaries H-G and F-E were taken as linear relationships between the extrapolated values H'-G' and F'-E', respectively. These boundary relationships are evident in Figure 27.

Figure 26 shows the smooth surfaces formed by the stress function values at the boundary lines. An extension of these surfaces, to include the stress function values within the boundaries, yields a smooth undulating surface above and below the plane of the plate.
Figure 26. Boundary Stress Function Surfaces
Plate C
Figure 27. Stress Function, $\phi$, For Pure Compression Plate C
Figure 28. Stress, $\sigma_x$, For Pure Compression Plate C
Figure 29. Stress, \( \sigma_y \), For Pure Compression Plate C
Figure 30. Shearing Stress, $\tau_{xy}$, For Pure Compression
Figure 31. Principal Stress Orientation, $\sigma$, For Pure Compression
Plate C
Figure 32. Maximum Principal Stress, $\sigma_p$, For Pure Compression Plate C.
EXAMPLE FOUR

THE HALF PLATE MOMENT PROBLEM

PLATE D

EXAMPLE BOUNDARY STRESS FUNCTION CALCULATIONS

AND

COMPUTED STRESS DISTRIBUTION DATA
EXAMPLE BOUNDARY STRESS FUNCTION CALCULATIONS

Consider the half plate section, dimensioned in terms of the mesh spacing "h" as shown in Figure 33. The mesh system employed is shown in Figure 25.

Plate D is subject, as was Plate B, to a pure moment stress condition. Plate D has the same general boundary outlines as Plate B. The basic boundary equations, determined for Plate B, will be employed here as needed.

Substituting into equation (32),

\[ \phi_n = \pm \frac{25}{9} \phi_n \]

where the neutral surface is at the center of boundary C-E.

Employing equation (41) to calculate stress function values along the boundary A-B, where the constant \( K_1 \) is taken equal to unity:

<table>
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<th>( \phi )</th>
<th>( x )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>1</td>
<td>1</td>
<td>7</td>
<td>343</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>8</td>
<td>512</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
<td>9</td>
<td>729</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
<td>10</td>
<td>1000  Pt. B</td>
</tr>
<tr>
<td>5</td>
<td>125</td>
<td>11</td>
<td>1331  Pt. B'</td>
</tr>
</tbody>
</table>

For the surface A-H, the above values are equal but opposite in sign.

From equations (38) and (39), if \( K_1 = 1 \), then \( K_2 = 3 \) and \( \phi_n = 60 \).
Figure 33.
Boundary Stress Relationship
Plate D
At point C, from equations (40a) and (41a),

\[ \frac{\partial \phi}{\partial y} = K_2 R^2 = 300 \]

\[ \phi = K_1 R^3 = 1000 \]

and the stress function values remain constant at 1000 along the surface B-C.

At point G, from equations (40c) and (41b),

\[ \frac{\partial \phi}{\partial y} = -K_2 R^2 = -300 \]

\[ \phi = -K_1 R^3 = -1000 \]

and the stress function values remain constant at -1000 along the surface H-G.

For the surface G-F, employing equation (43a):

<table>
<thead>
<tr>
<th>Pt. G'</th>
<th>( y )</th>
<th>( \phi )</th>
<th>Pt. F'</th>
<th>( y )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
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<td>6</td>
<td>200</td>
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<tr>
<td>9</td>
<td>-700</td>
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<td>-100</td>
<td>2</td>
<td>1400</td>
<td>Pt. F</td>
<td></td>
</tr>
</tbody>
</table>

At points F and E, from equations (40e) and (44a),

\[ \frac{\partial \phi}{\partial y} = -K_2 R^2 = -300 \]

\[ \phi = -K_2 R^2 w + R^3 (K_2 - K_1) = 1400 \]
and the stress function values remain constant at 1400 along the surface F-E.

For the surface D-C, as the boundary condition is positive,

\[ \frac{\partial \phi}{\partial s} = \frac{K_2 K_4 \alpha_m s^2}{K_6} + C_5 \]  

and

\[ \phi = \frac{K_1 K_3 \alpha_m s^3}{K_5} + C_5 s + C_6 \]  

From equation (48), as \( K_1 = 1 \), \( \alpha_n = 60 \) and \( R = 10 \),

\[ K_5 = 60 \]

From equation (49), as \( K_2 = 3 \), \( \alpha_n = 60 \) and \( R = 10 \),

\[ K_4 = 20 \]

Similarly, from equations (50) and (51),

\[ K_5 = 36 \]

and

\[ K_6 = 12 \]

Therefore, along the boundary D-C,

\[ \frac{\partial \phi}{\partial s} = \frac{125}{9} s^2 + C_5 \]  

and

\[ \phi = \frac{125}{27} s^3 + C_5 s + C_6 \]  

But, for point C (s = T = 6),

\[ \frac{\partial \phi}{\partial y} = 300 \]
and

\[ \phi = 1000 \]

so

\[ c_5 = -200 \]

and

\[ c_6 = 1200 \]

Therefore, for the surface D-C,

\[ \frac{\partial \phi}{\partial s} = \frac{125}{9} s^2 - 200 \]  

(46c)

and

\[ \phi = \frac{125}{27} s^3 - 200 s + 1200 \]  

(47c)

Substitution of \( s \) values into equation (47c), for the surface D-C, it is evident that:

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \phi )</th>
<th>( s )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pt. D</td>
<td>0</td>
<td>1200</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1005</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>837</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>725</td>
<td>7</td>
</tr>
</tbody>
</table>

Along the surface D-E, as the boundary condition is negative,

\[ \frac{\partial \phi}{\partial s} = -\frac{125}{9} s^2 + c_5 \]  

(46b)

At point E,

\[ \frac{\partial \phi}{\partial y} = -300 \]
and

\[ s = 6 \]

so

\[ C_5 = 200 \]

Then, along the surface D-E,

\[ \phi = -\frac{125}{27} s^3 + 200 s + C_6 \]

At point D, \( s = 0 \) and \( \phi = 1200 \), so

\[ C_6 = 1200 \]

Then, along the surface D-C,

\[ \phi = -\frac{125}{27} s^3 + 200 s + 1200 \quad (47d) \]

Substitution of \( s \) values into equation (47d), for the surface D-E, it is evident that:

<table>
<thead>
<tr>
<th>Pt. D</th>
<th>( s )</th>
<th>( \phi )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1200</td>
<td>1704</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1395</td>
<td>1621</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1563</td>
<td>1400</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1675</td>
<td>1012</td>
</tr>
</tbody>
</table>

Pt. E  
Pt. E'

The above completes the boundary stress function value calculations. From this point, the procedure is the same as for the previous problems. That is, extrapolation of the stress function values immediately outside the boundaries, using the method developed earlier, for Plate C. The initial stress function approximations
within the boundaries are then determined and the boundary values and initial approximations are taken together as computer input. The computed data, for Plate D, is shown on the succeeding pages.
Figure 34. Stress Function, \( \phi \), For Pure Bending Plate D
**Figure 35. Stress, \( \sigma \), For Pure Bending**

Plate D

![Stress Diagram](image-url)
**Figure 36. Stress, \( \sigma_y \), For Pure Bending**  
Plate D
Figure 37. Shearing Stress, \( \tau_{xy} \), For Pure Bending Plate D
Figure 38. Principal Stress Orientation, $\sigma$, For Pure Bending Plate D
Figure 39. Maximum Principal Stress, $\sigma_p$, For Pure Bending
Plate D
APPENDIX B

THE FULL PLATE COMPUTER PROGRAMS

AND

FLOW DIAGRAMS

THE HALF PLATE COMPUTER PROGRAMS
PASS ONE OF THE FULL PLATE PROBLEM

A FORTRAN II PROGRAM FOR SATISFYING THE
BIHARMONIC MOLECULE AND COMPUTING SIGX, SIGY AND
TXY AS APPLIED TO A RECTANGULARLY NOTCHED
PLATE SUBJECT TO VARIOUS BOUNDARY CONDITIONS.

1 FORMAT(1013)
2 FORMAT(1313)
17 FORMAT(1113)
3 FORMAT(8F10.4/5F10.4)
4 FORMAT (F10.4,F10.4)
5 FORMAT (8F10.4/3F10.4,13)
16 FORMAT (8F10.4/5F10.4,13)

DIMENSION F(23,13),SIGX(23,13)
DIMENSION SIGY(23,13),TXY(23,13)
DIMENSION M(25),N(25)

CLEAR THE ARRAYS
6 DO 7 I=1,23
   DO 7 J=1,13
   F(I,J)=0.0
7 FORMAT(1013)

READ LIMITS ON BIHARMONIC AND STRESS SWEEPS
READ 2,M1,M2,N1,N2,M6,N3,M8,N8,M9,N9,M21,M22,N18
READ 1,M11,M12,M13,M14,M15,M16,M17,M18,M19,M20
READ 17,M4,M5,N4,N5,N11,N12,N13,N14,N15,N16,N17

READ FIRST APPROXIMATIONS (F)
8 READ 3, ((F(I,J)),J=1,N3),I=1,M6)
ICRD=-1

START AN ITERATION - SWEEP THE BIHARMONIC MOLECULE
C TEST EQUALS LARGEST ERROR VALUE
C DCT EQUALS LARGEST ERROR VALUE DECIMAL
10 N6=0
11 DCT = 0.0
12 TEST = 0.0
C ELEM EQUALS COMPUTED F VALUES AT NODAL POINTS
13 DO 29 I=M1,M2
   DO 29 J=N1,N2
15 ELEM=(8.0*(F(I+1,J)+F(I,J+1)+F(I-1,J)+F(I,J-1))
   -2.0*(F(I+1,J+1)+F(I+1,J-1)+F(I-1,J+1)+F(I-1,J-1))
   -C-1.0*(F(I,J+2)+F(I+2,J)+F(I,J-2)+F(I-2,J)))/20.
C CHECK ERROR: ECK MEANS ERROR CHECK
ECK=ABSF(F(I,J)-ELEM)
IF(ECK-TEST)25,23,23

(CONTINUED)
23 TEST=ECK
     C DECK MEANS DECIMAL CHECK
25 DECK = ABSF(ECK/ELEM)
     IF (DECK-DCT)29,27,27
27 DCT=DECK
29 F(I,J)=ELEM
     C END OF ONE NODAL POINT ITERATION
31 CONTINUE
     PRINT 4,TEST,DCT
33 CONTINUE
     IF(N6-1)34,37,71
34 M1=M1+M11
     M2=M2+M12
     N1=N1+M18
     N2=N2+M13
35 N6=N6+1
     GO TO 11
37 M1=M1+M14
     M2=M2+M15
     N1=N1+M19
     N2=N2+M20
     N6=N6+1
     GO TO 11
71 M1=M1+M16
     M2=M2+M17
     N1=N1+M21
     N2=N2+M22
     IF(SENSE SWITCH 1)73,10
73 PUNCH 16*((F(I,J),J=1,N3),ICRD,I=1,M6)
     IF (SENSE SWITCH 2)75,10
75 CONTINUE
     DO 77 I=1,M6
     DO 77 J=1,N3
     SIGY(I,J)=0.0
     TXY(I,J)=0.0
77 SIGX(I,J)=0.0
     N7=0
     C SWEEP THE STRESS MOLECULES
79 DO 81 I=M4,M5
     DO 81 J=N4,N5
     SIGX(I,J)=(F(I,J-1)+F(I,J+1)-2.*F(I,J))/30.
     SIGY(I,J)=(F(I-1,J)+F(I+1,J)-2.*F(I,J))/30.
     TXY(I,J)=(F(I-1,J-1)+F(I+1,J+1)-F(I+1,J-1)
     C F(I-1,J+1))/120.
82 CONTINUE

(CONTINUED)
IF (N7-1)95,105,115
95 M4=M4+N11
M5=M5+N12
N4=N4+N16
N5=N5+N13
N7=N7+1
GO TO 79
105 M4=M4+N14
M5=M5+N15
N4=N4+N17
N5=N5+N18
N7=N7+1
GO TO 79
115 PUNCH 5, ((SIGX(I,J), J=N8,N9), ICRD, I=M8,M9)
PAUSE
PUNCH 5, ((SIGY(I,J), J=N8,N9), ICRD, I=M8,M9)
PAUSE
PUNCH 5, ((TXY(I,J), J=N8,N9), ICRD, I=M8,M9)
M4=2
M5=8
N4=2
N5=12
GO TO 10
END

PASS TWO OF THE FULL PLATE PROBLEM

1 FORMAT (8F10.4/3F10.4)
3 FORMAT (8F10.4/3F10.4,13)
5 FORMAT (14I3)
DIMENSION SIGX(21,11),SIGY(21,11),TXY(21,11)
DIMENSION SIGP1(21,11),SIGP2(21,11),ANGL(21,11)
DIMENSION M(15),N(15),BIG(21,11)
6 READ 5,N1,M1,M4,M5,N4,N5,M8,M9,M10,M11,N8,N9,N10,N11
7 READ 1, ((SIGX(I,J), J=1,N1), I=1,M1)
9 READ 1, ((SIGY(I,J), J=1,N1), I=1,M1)
11 READ 1, ((TXY(I,J), J=1,N1), I=1,M1)
ICRD=-1
13 DO 15 I=1,M1
DO 15 J=1,N1
SIGP1(I,J)=0.0
(CONTINUED)
ANGL(I,J)=0.0
BIG(I,J)=0.0

15 SIGP2(I,J)=0.0
  N7=0

17 DO 20 I=M4,M5
  DO 20 J=N4,N5
  ANGL(I,J)=ATANF(-TXY(I,J)*2./SIGX(I,J))
  C=SIGY(I,J))*28.648
  D=SQRTF((((SIGX(I,J)-SIGY(I,J))/2.))*2+TXY(I,J)**2)
  SP=(SIGX(I,J)+SIGY(I,J))/2.
  SIGP1(I,J)=SP+D
  SIGP2(I,J)=SP-D
  IF(ABSF(SIGP1(I,J))-ABSF(SIGP2(I,J)))31,32,19
C BIG IS THE LARGEST NUMERICAL VALUE OF P1 AND
C P2 AT ANY ONE NODAL POINT
31 BIG(I,J)=SIGP2(I,J)
  GO TO 20
32 BIG(I,J)=SIGP1(I,J)
  GO TO 20
19 BIG(I,J)=SIGP1(I,J)
20 N12=0
21 CONTINUE
  IF(N7-1)23,25,27
23 M4=M4+M8
  M5=M5+M9
  N4=N4+M10
  N5=N5+M11
  N7=N7+1
  GO TO 17
25 M4=M4+N8
  M5=M5+N9
  N4=N4+N10
  N5=N5+N11
  N7=N7+1
  GO TO 17
27 PUNCH 3, ((ANGL(I,J),J=1,N1),ICRD,I=1,M1)
  PAUSE
  PUNCH 3, ((SIGP1(I,J),J=1,N1),ICRD,I=1,M1)
  PAUSE
  PUNCH 3, ((SIGP2(I,J),J=1,N1),ICRD,I=1,M1)
  PAUSE
  PUNCH 3, ((BIG(I,J),J=1,N1),ICRD,I=1,M1)
END
FLOW CHART- FULL PLATE-PASS ONE

Start

Dimensions ➔ Formats ➔ Clear φ Arrays

Compute Biharmonic Molecule (1 Nodal Pt.) ➔ Initial φ ➔ Sweep Constants

Compare φi and φc

Discard Error ➔ Error Test ➔ Yes ➔ Keep Error ➔ Test = Error

Begin New Iteration Cycle

Compute Decimal Chk.

Discard Deck ➔ Deck Decl. ➔ Yes ➔ Keep Deck

Iterate Next Nodal Point

Keep Computed φ Value

No ➔ Plate Section Iterated?

Print Error and Decimal Error

Entire Plate Iterated?

First Section Sweep Limits ➔ Yes

(Continued)
Is Sense Switch 1 On?

No

Compute Plate Section Stresses ($\sigma_x, \sigma_y, \tau_{xy}$)

Yes

Is Sense Switch 2 On?

No

Clear Stress Arrays ($\sigma_x, \sigma_y, \tau_{xy}$)

Yes

All Plate Stresses Computed?

Yes

Punch Stresses ($\sigma_x, \sigma_y, \tau_{xy}$)

No

Next Plate Section Sweep Limits

Are Stresses O.K.?

No

Instant Stop Switch - Clear Computer

Yes

$\sigma_x, \sigma_y$ and $\tau_{xy}$ To Pass Two Program

End
FLOW CHART—FULL PLATE—PASS TWO

Start → Formats → Dimensions → Sweep Constants

Clear $\theta, \sigma_1, \sigma_2$ and $\sigma_p$ Arrays → $\sigma_x, \sigma_y, T_{xy}$ From Pass One

Compute $\theta, \sigma_1, \sigma_2$ and $\sigma_p$ For One Nodal Point

Plate Section Computed?

Yes → Entire Plate Computed?

Yes → Punch $\theta, \sigma_1, \sigma_2$ and $\sigma_p$

End

No → Compute Next Nodal Pt.

No → Next Section Sweep Limits

No → Plate Section Computed?

Yes → Entire Plate Computed?

Yes → Punch $\theta, \sigma_1, \sigma_2$ and $\sigma_p$

End
PASS ONE OF THE HALF PLATE PROBLEM

A FORTRAN II PROGRAM FOR SATISFYING THE
BIHARMONIC MOLECULE AND COMPUTING SIGX, SIGY
AND TXY AS APPLIED TO A RECTANGULARLY NOTCHED
PLATE SUBJECT TO VARIOUS BOUNDARY CONDITIONS.

1 FORMAT(813)
2 FORMAT(1013)
3 FORMAT(12F6.0/11F6.0)
4 FORMAT(F10.4,F10.4)
5 FORMAT(11F6.2/10F6.2,12)

DIAMETER F(23,23)*SUBST(23,23)
DIMENSION M(20),N(15)

CLEAR THE ARRAYS

DO 7 I=1,23
DO 7 J=1,23

F EQUALS STRESS FUNCTION AT NODAL POINTS

F(I,J)=0.0

READ LIMITS ON BIHARMONIC AND STRESS SWEEPS
READ 2,M1,M2,N1,N2,M6,N3,M8,N8,M9,N9
READ 1,M11,M12,M13,M14,M15,M16,M17,M18
READ 1,M4,M5,N4,N5,N11,N12,N13,N14

READ FIRST APPROXIMATIONS (F)

READ 3, ((F(I,J),J=1,N3),I=1,M6)

START AN ITERATION - SWEEP THE BIHARMONIC MOLECULE
TEST EQUALS LARGEST ERROR VALUE
DCT EQUALS LARGEST ERROR VALUE DECIMAL

TEST = 0.0

ELEM EQUALS COMPUTED F VALUES AT NODAL POINTS
DCT=0.0

DO 29 I=M1,M2
DO 29 J=N1,N2

ELEM=(8.*(F(I+1,J)+F(I,J+1)+F(I-1,J)+F(I,J-1))
C-2.*(F(I+1,J+1)+F(I+1,J-1)+F(I-1,J+1)+F(I-1,J-1))
C-1.*(F(I,J+2)+F(I+2,J)+F(I,J-2)+F(I-2,J)))/20.

CHECK ERROR, ECK MEANS ERROR CHECK
ECK=ABSF(F(I,J)-ELEM)
IF(ECK-TEST)25,23,23

TEST=ECK

DECK MEANS DECIMAL CHECK

(CONTINUED)
IF(DECK = DCT) 29, 27, 27
27 DCT = DECK
29 F(I, J) = ELEM
C END OF ONE NODAL POINT ITERATION
   IF(SENSE SWITCH 3) 32, 33
32 PRINT 4, TEST, DCT
33 IF(N6 = 1) 37, 34, 37
34 M1 = M1 + M11
   M2 = M2 + M12
   N1 = N1 + M18
   N2 = N2 + M13
35 N6 = N6 + 1
   GO TO 12
37 M1 = M1 + M14
   M2 = M2 + M15
   N1 = N1 + M16
   N2 = N2 + M17
   IF(SENSE SWITCH 1) 73, 10
73 PUNCH 16, ((F(I, J), J = 1..N3), ICRD, I = 1..M6)
   IF(SENSE SWITCH 2) 75, 10
75 DO 77 I = 1, M6
   DO 77 J = 1, N3
77 SUBST(I, J) = 0.0
C SWEEP THE STRESS MOLECULES
79 DO 81 I = M4, M5
   DO 81 J = N4, N5
C SUBST IS ACTUALLY SIGX
81 SUBST(I, J) = ((F(I, J) - 1) + F(I, J) + 1 - 2 * F(I, J)) / 60.
   IF(N7 = 1) 85, 83, 85
83 M4 = M4 + N11
   M5 = M5 + N12
   N4 = N4 + N14
   N5 = N5 + N13
   N7 = N7 + 1
   GO TO 79
85 PUNCH 5, ((SUBST(I, J), J = N8, N9), ICRD, I = M8, M9)
   M4 = M4 - N11
   M5 = M5 - N12
   N4 = N4 - N14
   N5 = N5 - N13
   N7 = 1
87 DO 89 I = M4, M5
   DO 89 J = N4, N5
C SUBST IS ACTUALLY SIGY

(CONTINUED)
89 SUBST(I,J) = (F(I-1,J) + F(I+1,J) - 2*F(I,J))/60.
   IF(N7-1) 93, 91, 93
91 M4=M4+N11
   M5=M5+N12
   N4=N4+N14
   N5=N5+N13
   N7=N7+1
   GO TO 87
93 PUNCH 5, ((SUBST(I,J), J=N8,N9), ICRD, I=M8,M9)
   M4=M4-N11
   M5=M5-N12
   N4=N4-N14
   N5=N5-N13
   N7=1
95 DO 97 I=M4,M5
   DO 97 J=N4,N5
   SUBST IS ACTUALLY TXY
97 SUBST(I,J) = (F(I-1,J-1) + F(I+1,J+1) - F(I+1,J-1)
   C-F(I-1,J+1))/240.
   IF(N7-1) 101, 99, 101
99 M4=M4+N11
   M5=M5+N12
   N4=N4+N14
   N5=N5+N13
   N7=N7+1
   GO TO 95
101 PUNCH 5, ((SUBST(I,J), J=N8,N9), ICRD, I=M8,M9)
   M4=M4-N11
   M5=M5-N12
   N4=N4-N14
   N5=N5-N13
   GO TO 10
END

PASS TWO OF THE HALF PLATE PROBLEM

1 FORMAT(11F6.2/10F6.2)
2 FORMAT(11F6.1/10F6.1, I2)
3 FORMAT(11F6.2/10F6.2, I2)
5 FORMAT(10I3)
   DIMENSION SIGX(11,21), SIGY(11,21), TXY(11,21)
   DIMENSION SIGP1(11,21), SIGP2(11,21), ANGL(11,21)

(CONTINUED)
DIMENSION M(12),N(10),BIG(11,21)
6 READ 5,N1,M1,M4,M5,N4,N5,M8,M9,M10,M11
7 READ 1,((SIGX(I,J),J=1,N1),I=1,M1)
9 READ 1,((SIGY(I,J),J=1,N1),I=1,M1)
11 READ 1,((TXY(I,J),J=1,N1),I=1,M1)
ICRD=-1
13 DO 15 I=1,M1
  DO 15 J=1,N1
  SIGP1(I,J)=0.0
  ANGL(I,J)=0.0
  BIG(I,J)=0.0
15 SIGP2(I,J)=0.0
N7=1
17 DO 20 I=M4,M5
  DO 20 J=N4,N5
  ANGL(I,J)=ATANF(-TXY(I,J)*2./(SIGX(I,J))
  SIGY(I,J))**28.648
  D=SQRTF((SIGX(I,J)-SIGY(I,J))/2.*TXY(I,J)**2)
  SP=(SIGX(I,J)+SIGY(I,J))/2.
  SIGP1(I,J)=SP+D
  SIGP2(I,J)=SP-D
31 BIG(I,J)=ABSF(SIGP1(I,J))-ABSF(SIGP2(I,J))31,32,19
C BIG IS THE LARGEST NUMERICAL VALUE OF P1
C AND P2 AT ANY ONE NODAL POINT
31 BIG(I,J)=SIGP2(I,J)
  GO TO 20
32 BIG(I,J)=SIGP1(I,J)
  GO TO 20
19 BIG(I,J)=SIGP1(I,J)
20 N10=0
21 CONTINUE
  IF(N7=1)27,23,27
23 M4=M4+M8
  M5=M5+M9
  N4=N4+M10
  N5=N5+M11
  N7=N7+1
  GO TO 17
27 PUNCH 2,((ANGL(I,J),J=1,N1),ICRD,I=1,M1)
  PAUSE
  PUNCH 3,((SIGP1(I,J),J=1,N1),ICRD,I=1,M1)
  PAUSE
  PUNCH 3,((SIGP2(I,J),J=1,N1),ICRD,I=1,M1)
  PAUSE
  PUNCH 3,((BIG(I,J),J=1,N1),ICRD,I=1,M1)
  GO TO 6
END
APPENDIX C

EXAMPLE FINITE DIFFERENCE MOLECULE DERIVATIONS
EXAMPLE FINITE DIFFERENCE MOLECULE DERIVATIONS

This section illustrates the method of converting partial differential equations to finite difference operator molecules. The derivations presented here follow those of Timoshenko and Goodier.¹⁸

The partial surface of a thin plate, ABCD, as shown in Figure 40, is replaced by a mesh system of discrete nodal points. The nodal points are spaced an equal distance "h" in directions perpendicular to the X and Y axes. The positive stress function values are shown as ordinates originating at, and perpendicular to, the surface of the plate. When connected by a smooth curve at the top, the stress function values form a smooth surface above the plane of the plate.

Figure 40. Example Mesh System

Calculation of the first difference, the slope, yields:

\[
\frac{\partial \phi}{\partial y} \approx \frac{\phi_2 - \phi_0}{h}
\]

or

\[
\frac{\partial \phi}{\partial y} \approx \frac{\phi_2 - \phi_4}{2h}
\]

The equation immediately above can be replaced by the finite difference operator molecule:

The approximate value of the second difference can be calculated as:

\[
\frac{\partial^2 \phi}{\partial y^2} \approx \frac{(\phi_2 - \phi_0)}{h^2} - \frac{\phi_0 - \phi_4}{h^2} = \frac{\phi_2 - 2\phi_0 + \phi_4}{h^2}
\]

The second difference equation can be represented by the finite difference operator molecule:
Operator molecules can be derived for expressions of a higher order by employing a similar procedure.
APPENDIX D

NOTATION
NOTATION

$\phi$ = the stress function.

$x$ = a variable distance along the X-axis.

$y$ = a variable distance along the Y-axis.

$s$ = a variable distance along the Y-axis.

$h$ = the spacing dimension for a square mesh.

$T$, $R$ and $W$ = variable distances parallel to the Y-axis.

$\sigma_x$ = a stress parallel to the X-axis.

$\sigma_y$ = a stress parallel to the Y-axis.

$\tau_{xy}$ = a shear couple at a point.

$\sigma_n$ = a normalized $\sigma_x$.

$\sigma_r$ = a normalized $\sigma_x$.

$\sigma_u$ = a calculated $\sigma_x$.

$\sigma_1$ = major principal stress.

$\sigma_2$ = minor principal stress.

$\sigma_p$ = maximum principal stress.

$\phi$ = the angular orientation of principal stress planes with respect to the X-axis.

$C$ = a constant of integration.

$K$ = a constant.