Analysis of Heat Transfer from Circular or Square Fins

Yun-Seng Huang

Follow this and additional works at: https://openprairie.sdstate.edu/etd

Recommended Citation
Huang, Yun-Seng, "Analysis of Heat Transfer from Circular or Square Fins" (1967). Electronic Theses and Dissertations. 3307.
https://openprairie.sdstate.edu/etd/3307
ANALYSIS OF HEAT TRANSFER FROM CIRCULAR OR SQUARE FINS

BY

YUN-SENG HUANG

A thesis submitted in partial fulfillment of the requirements for the degree Master of Science, Major in Mechanical Engineering, South Dakota State University

1967

SOUTH DAKOTA STATE UNIVERSITY LIBRARY
ANALYSIS OF HEAT TRANSFER FROM CIRCULAR OR SQUARE FINS

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser

Head, Mechanical Engineering Department
ACKNOWLEDGEMENT

The author is indebted to Professor B. E. Eno for his valuable cooperation and guidance throughout the research and also his great assistance in the writing of the thesis.

· YSH
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>3</td>
</tr>
<tr>
<td>II. ANALYSIS OF HEAT TRANSFER FROM CIRCULAR FINS OF CONSTANT THICKNESS</td>
<td>6</td>
</tr>
<tr>
<td>A. Statement of the Problem</td>
<td>6</td>
</tr>
<tr>
<td>B. The Solution</td>
<td>8</td>
</tr>
<tr>
<td>C. Sample Problem</td>
<td>17</td>
</tr>
<tr>
<td>D. Discussion</td>
<td>19</td>
</tr>
<tr>
<td>III. ANALYSIS OF HEAT TRANSFER FROM SQUARE FINS OF CONSTANT THICKNESS</td>
<td>30</td>
</tr>
<tr>
<td>A. Statement of the Problem</td>
<td>30</td>
</tr>
<tr>
<td>B. The Solution</td>
<td>30</td>
</tr>
<tr>
<td>C. Sample Problem</td>
<td>44</td>
</tr>
<tr>
<td>D. Discussion</td>
<td>47</td>
</tr>
<tr>
<td>IV. CONCLUSIONS AND RECOMMENDATIONS</td>
<td>52</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>54</td>
</tr>
<tr>
<td>APPENDIX</td>
<td>55</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>1.</td>
<td>Circular fin in r, ( \phi ) coordinate system</td>
</tr>
<tr>
<td>2.</td>
<td>The dimensionless temperature distribution at ( \tilde{r} = 1 ) for different ( \phi )</td>
</tr>
<tr>
<td>3.</td>
<td>Temperature distribution for variable B for ( r_i/r_0 = 0.6, \theta_a = \theta_e = 0.5, C=1, n=1/4 )</td>
</tr>
<tr>
<td>4.</td>
<td>Temperature distribution for variable B for ( r_i/r_0 = 0.6, \theta_a = \theta_e = 0.7, C=1, n=1/4 )</td>
</tr>
<tr>
<td>5.</td>
<td>Temperature distribution for variable B for ( r_i/r_0 = 0.6, \theta_a = \theta_e = 0.9, C=1, n=1/4 )</td>
</tr>
<tr>
<td>6.</td>
<td>Temperature distribution for variable B for ( r_i/r_0 = 0.4, \theta_a = \theta_e = 0.5, C=1, n=1/4 )</td>
</tr>
<tr>
<td>7.</td>
<td>Temperature distribution for variable B for ( r_i/r_0 = 0.4, \theta_a = \theta_e = 0.7, C=1, n=1/4 )</td>
</tr>
<tr>
<td>8.</td>
<td>Temperature distribution for variable B for ( r_i/r_0 = 0.4, \theta_a = \theta_e = 0.9, C=1, n=1/4 )</td>
</tr>
<tr>
<td>9.</td>
<td>Temperature distribution for variable C for ( r_i/r_0 = 0.4, \theta_a = \theta_e = 0.7, B=2, n=1/4 )</td>
</tr>
<tr>
<td>10.</td>
<td>Square fin in r, ( \phi ) coordinate system</td>
</tr>
<tr>
<td>11.</td>
<td>Temperature distribution in square fin with ( p = 1, \theta_a = 0.7, ) and ( b/r_i = 2 )</td>
</tr>
<tr>
<td>12.</td>
<td>Percent error of adiabatic condition at ( x = b ) in second and third order solution</td>
</tr>
<tr>
<td>13.</td>
<td>Percent error of adiabatic condition at ( x = b ) in second order solution for different ( b/r_i )</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

It has been recognized that an increase in the rate of heat transfer to or from a solid body can be accomplished by extended surfaces called fins. There are a variety of fins such as rectilinear or parabolic fins protruding from a flat wall, or thin circular or rectangular fins on a tube. In all cases the heat transfer designers are interested in the optimum heat loss or gain. Considerable experimental work has been done in order to establish the empirical values of convective heat transfer coefficient to predict the amount of heat transfer from a particular fin configuration. The limitation of the experimental approach is that the result for each particular fin can not be applied widely to others.

An alternative way to meet the problem is by means of analytical theory. This constitutes treating the problem as a boundary value problem for which a governing differential equation is derived and appropriate boundary conditions are specified. The advantage of this approach is that the result can often be widely applied just by specification of certain geometric and physical parameters.

The purpose of this thesis is to find analytically the temperature distribution and heat loss of a circular fin of constant thickness on a tube and the temperature distribution and heat loss of a square fin of constant thickness on a tube. Some related investigations should be mentioned here. Eno[1]* found an approximate

* Bracketed numbers refer to the References.
solution for temperature distribution and heat loss from a rectilinear fin for combined convection and radiation. His analysis was based on a linearization of the second order differential equation which will be used in this thesis. The approximate solution proved to be rather good for most cases. Keller [2] considered the convection heat transfer for both axial and radial conduction in a circular fin on a tube and found the exact solution for temperature distribution and heat loss as a function of \( r \) and \( z \). Murray [3] dealt with a circular fin of constant thickness with a root temperature expressed as a function of angular distance from a reference plane and found an exact solution for temperature distribution and heat loss as a function of \( r \) and \( \phi \).

Schneider [4] gave some consideration to a \( 30^\circ \) symmetry nuclear reactor cell whose bounds were necessarily specified in both cartesian and polar coordinates. Uniform heat generation existed in the cell and Schneider found an approximate solution for the temperature distribution and heat loss by representing the whole boundary condition by only two points on one of the bounds in order to handle the mixed coordinates. Fend [5] found that the error for Schneider's problem would be only about three hundredths of one percent.

The analysis will be considered in two separate sections: first, the circular fin and second, the square fin. Individual discussions of the results of each analysis will be given at the end of each related section. A brief conclusion and recommendation follows.
NOMENCLATURE

A = Heat transfer area.

$A_n, A'_n =$ Arbitrary constants.

$B =$ Convection parameter given by Eq (11).

$b =$ One-half side dimension of square fin.

$B_n, B'_n =$ Arbitrary constants.

$C =$ Radiation parameter given by Eq (12).

$C_n, C'_n =$ Arbitrary constants.

$D =$ Characteristic fin dimension used in Eq (8).

$D_n =$ Arbitrary constant.

$f_i =$ Function of $\phi$ given by Eq (92).

$f(\phi) =$ Dimensionless root temperature of circular fin specified in Eq (14).

$g =$ Gravitational acceleration.

$G =$ Coefficient in empirical film coefficient correlations.

$Gr =$ Grashof number given in Eq (10).

$h =$ Film coefficient.

$\overline{h} =$ Uniform film coefficient.

$I_n =$ Modified Bessel function of the first kind.

$i =$ Positive integer.

$k_a =$ Conductivity of convective medium.

$k_f =$ Fin conductivity.

$k_n =$ Arbitrary constant.

$K_n =$ Modified Bessel function of the second kind.

$m =$ Exponent in empirical film coefficient correlations in Eqs (9) and (10).
M = Parameter given by Eq (22).

n = Exponent characteristic of forced or free convection used in Eq (8); eigenvalues in Eq (35) and (71).

Nu = Nusselt number used in Eq (8).

p = Parameter given by Eqs (21) and (55).

\bar{p} = Parameter given by Eq (82).

Pr = Prandtl number of convective medium.

q = Total fin heat loss.

\bar{q} = Dimensionless fin heat loss parameter defined by Eqs (45) and (94).

q_c = Heat transferred by convection.

q_k = Heat transferred by conduction.

q_R = Heat transferred by radiation.

r = Variable radial distance.

r_1 = Inner radius of circular or square fin.

\bar{r} = r/r_1.

r_0 = Outer radius of circular fin.

T = Variable temperature in the fin.

T_a = Convective medium temperature.

T_R = Radiant receiver temperature.

T_s = Fixed fin root temperature.

\bar{V} = Characteristic velocity of medium in forced convection used in Eq (9).

x = Variable distance in x direction.

\bar{x} = x/r_1.

x_1 = General coordinate.

y = Variable distance in y direction.
\[ \bar{y} = y/r_i. \]

\[ z = \text{Variable distance in } z \text{ direction.} \]

\[ \beta = \text{Expansion coefficient of convective medium.} \]

\[ \delta = \text{Thickness of the fin.} \]

\[ \Delta = \text{Determinant given by Eq (91).} \]

\[ \varepsilon = \text{Fin emissivity.} \]

\[ \eta = \text{Function of } \phi \text{ and } n \text{ defined by Eq (84).} \]

\[ \Theta = T/T_s. \]

\[ \Theta_a = T_a/T_s. \]

\[ \Theta_e = T_R/T_s. \]

\[ \bar{\Theta} = (1 - \Theta) \text{ in Eq (17); or } (\Theta - \Theta_a) \text{ in Eq (60).} \]

\[ \bar{\Theta}^* = (\bar{\Theta} + M) \text{ in Eq (23).} \]

\[ \nu = \text{Kinematic viscosity of convective medium.} \]

\[ \zeta = \text{Function of } \phi \text{ and } n \text{ defined by Eq (85).} \]

\[ \gamma = \text{Arbitrary constant used in Eq (47).} \]

\[ \sigma = \text{Boltzmann's radiation constant.} \]

\[ \Phi = \text{Angular direction in polar coordinate system.} \]

\[ \psi = \text{Function of } \bar{r} \text{ and } n \text{ defined by Eq (44).} \]
II. ANALYSIS OF HEAT TRANSFER FROM CIRCULAR FINS
OF CONSTANT THICKNESS

A. Statement of the Problem

The object is to find the equations expressing the temperature distribution and heat loss of a circular fin on a tube when heat is received through its root where temperature may be specified as a function of azimuthal angle and is lost to the surrounding medium from its exposed surfaces. In order to set up the differential equation of heat flow and to obtain a satisfactory solution, it will be necessary to make some assumptions and simplifications as follows:

1. The heat flow and temperature distribution through the fin are independent of time, i.e., steady state prevails.

2. The material of which the fin is composed is homogeneous and isotropic.

3. There are no heat sources in the fin.

4. All physical properties of the fin and convective medium remain constant.

5. The convective medium temperature and radiant receiver temperature are single predictable quantities, not necessarily equal.

6. The external surface resistance is so high compared to the internal resistance that the temperature gradient across the fin thickness may be neglected.

7. The heat transfer through the outermost edge of the fin at \( r = r_0 \) is negligible compared to that emitted by radiation and convection at the other exposed sides.

8. A plane of symmetry is assumed to exist as shown in Figure 1 at the bounds \( \phi = 0 \) and \( \phi = \pi \).

Gardner (6) showed that the errors involved in assumptions 6 and 7 are very small for most practical forms of extended surface.
Figure 1  Circular fin in r, r, \( \phi \) coordinate system.

Thickness = \( \delta \)
B. The Solution

There are three modes of heat transfer which are governed by

a. Fourier's conduction law

\[ q_k = -k_f A \frac{dT}{dx_i} \]  \hspace{1cm} (1)

b. Newton's cooling law

\[ q_c = h A (T - T_a) \]  \hspace{1cm} (2)

c. Stefan-Boltzman law

\[ q_R = \varepsilon A (T^4 - T_R^4) \]  \hspace{1cm} (3)

where the radiation shape factor has been assumed as unity.

The differential equation of heat flow may be set up by making a heat balance on the infinitesimal polar element of the fin as shown in Figure 1. This yields

\[ (q_k)_r + (q_k)_\phi = (q_k)_r + (q_k)_\phi + q_c + q_R \]  \hspace{1cm} (4)

Using the three laws stated by Eqs (1), (2), and (3), Eq (4) will yield

\[ \frac{\delta^2 T}{\delta r^2} + \frac{1}{r} \frac{\delta T}{\delta r} + \frac{1}{r^2} \frac{\delta^2 T}{\delta \phi^2} = \frac{2 \varepsilon}{k_f \delta} (T - T_a) + \frac{2 \delta \varepsilon}{k_f \delta} (T^4 - T_R^4) \]  \hspace{1cm} (5)

To non-dimensionalize, let

\[ \theta = \frac{T}{T_s} \quad \theta_a = \frac{T_a}{T_s} \quad \theta_e = \frac{T_R}{T_s} \quad \bar{T} = \frac{r}{k_i} \]  \hspace{1cm} (6)

Substituting these into Eq (5), we get a dimensionless partial
The variable film coefficient may be written in terms of the Nusselt number \([1,7]\) such that

\[
h = \frac{k_a N_u}{D} \left( \frac{\theta - \theta_a}{1 - \theta_a} \right)^n
\]

(8)

where for forced convection

\[
N_u = \text{Gr} (Re)^n \quad n = 0 \quad \text{Re} = \frac{\bar{V} D}{v}
\]

(9)

and for free convection

\[
N_u = \text{Gr} (Gr Pr)^n \quad n = m \quad \text{Gr} = \frac{g \beta (T_s - T_a) D^3}{\nu^2}
\]

(10)

Note that both \(N_u\) and \(\text{Gr}\) in Eq (10) are defined in terms of fixed temperatures \(T_s\) and \(T_a\) only, whereas the coefficient \(h\) in Eq (8) may be a function of the variable temperature \(T\) in the case of free convection.

Substituting for the film coefficient from Eq (8) into Eq (7) and letting

\[
B = \frac{2 R^2 k_a N_u}{k_f \delta D (1 - \theta_a)^n}
\]

(11)

and

\[
C = \frac{3 R^2 \gamma \epsilon T_s^3}{k_f \delta}
\]

(12)

equation (7) becomes

\[
\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{1}{R^2} \frac{\partial^2 \theta}{\partial \phi^2} = B (\theta - \theta_a)^{1+n} + \frac{C}{4} (\theta^4 - \theta_a^4)
\]

(13)
In the case of specifying a fixed film coefficient \( h \), this \( h \) may be substituted for \( \frac{Nu K_a}{D} \) in Eq (11) with \( n = 0 \).

The boundary conditions of Eq (13) will be determined according to the assumptions in the statement of the problem.

a. At the base of the fin

\[
\theta(\bar{r} = 1, \phi) = f(\phi) \tag{14}
\]

b. At the tip of the fin

\[
\frac{\partial \theta}{\partial \bar{r}}(\bar{r} = \bar{r}_o, \phi) = 0 \tag{15}
\]

c. From the assumption of the condition of symmetry, there can be no heat flow through the plane of symmetry; hence

\[
\frac{\partial \theta}{\partial \phi}(\bar{r}, \phi = 0) = \frac{\partial \theta}{\partial \phi}(\bar{r}, \phi = \pi) = 0 \tag{16}
\]

Since all temperatures, even \( T_a \) and \( T_R \), are absolute values, we would expect \( T_a \) and \( T_R \) not to be very different from \( T_s \). Thus we make the substitution

\[
\theta = 1 + \bar{\theta} \tag{17}
\]

where we assume \( |\bar{\theta}| \ll 1 \) for most practical cases. We see that a linear approximation will yield

\[
\theta^4 = (1 + \bar{\theta})^4 = 1 + 4\bar{\theta} + 6\bar{\theta}^2 + \cdots \approx 1 + 4\bar{\theta} \tag{18}
\]

and
Substituting Eqs (17), (18), and (19) into Eq (13), we get
\[
(\theta - \theta_0)^{i+n} = [(1 - \theta_0) + \bar{\theta}]^{i+n} \\
= (1 - \theta_0)^{i+n} + (1+n)(1-\theta_0)^n\bar{\theta} + \frac{(1+n)n}{2!}(1-\theta_0)^{n-1}\bar{\theta}^2 + \ldots
\]
(19)

where we have set
\[
p^2 = B(1+n)(1-\theta_0)^n + C
\]
(20)

and
\[
M = \frac{B(1-\theta_0)^{i+n} + \frac{C}{p^2}(1-\theta_e^* \epsilon)}{p^2}
\]
(22)

Letting
\[
\bar{\bar{\theta}}^* = \bar{\theta} + M
\]
(23)

equation (20) becomes
\[
\frac{\partial^2 \bar{\bar{\theta}}^*}{\partial \bar{\bar{r}}^2} + \frac{\partial \bar{\bar{\theta}}^*}{\partial \bar{\bar{r}}} + \frac{1}{\bar{\bar{r}}^2} \frac{\partial \bar{\bar{\theta}}^*}{\partial \phi^2} - P \bar{\bar{\theta}}^* = 0
\]
(24)

Using the method of separation of variables, we assume a product solution
\[
\bar{\bar{\theta}}^* = \bar{\bar{R}}(\bar{\bar{r}}) \bar{\bar{\Phi}}(\phi)
\]
(25)

which may reduce Eq (24) to
\[
\frac{1}{R} \left[ \bar{R}^2 \frac{d^2 \bar{R}}{d\bar{r}^2} + \bar{R} \frac{d\bar{R}}{d\bar{r}} - p^2 \bar{R}^3 \right] = - \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = k_n^2
\]  

(26)

where either side of the equation, being independent, allows us to assign the constant \( k_n^2 \).

Eq (26) may give the two ordinary differential equations

\[
\bar{R}^2 \frac{d^2 \bar{R}}{d\bar{r}^2} + \bar{R} \frac{d\bar{R}}{d\bar{r}} - (k_n^2 + p^2 \bar{R}^2) \bar{R} = 0
\]

(27)

and

\[
\frac{d^2 \Phi}{d\phi^2} + k_n^2 \Phi = 0
\]

(28)

Eq (27) is a modified Bessel equation. The general solutions for Eqs (27) and (28) are

\[
\bar{R} = C_n' I_{k_n}(p\bar{R}) + D_n' K_{k_n}(p\bar{R})
\]

(29)

and

\[
\Phi = A_n' \cos k_n \phi + B_n' \sin k_n \phi
\]

(30)

Substituting Eqs (29) and (30) into Eq (25) and using the definitive Eqs (17) and (23), the general solution becomes

\[
\Theta = 1 - M + \sum_{n=0}^{\infty} (A_n \cos k_n \phi + B_n \sin k_n \phi) [I_{k_n}(p\bar{R}) + C_n K_{k_n}(p\bar{R})]
\]

(31)

where \( A_n, B_n, C_n, \) and \( k_n \) are arbitrary constants which have absorbed \( c_n' \).

Applying the boundary conditions of Eq (16) to Eq (31)
\[ \frac{\partial \Theta}{\partial \Phi} (r, \phi = 0) = 0 = \sum_{n=0}^\infty B_n K_n \left[ I_{kn}(p r) + C_n K_{kn}(p r) \right] \] (32)

we set

\[ B_n = 0 \] (33)

and

\[ \frac{\partial \Theta}{\partial \Phi} (r, \phi = \pi) = 0 = - \sum_{n=0}^\infty A_n K_n \sin kn \pi \left[ I_{kn}(p r) + C_n K_{kn}(p r) \right] \] (34)

where we can only conclude that

\[ \sin kn \pi = 0 \]

hence

\[ k_n = n \quad (n = 0, 1, 2, 3, \ldots) \] (35)

Substituting Eqs (33) and (35) into Eq (31), we get

\[ \Theta = 1 - M + \sum_{n=0}^{\infty} A_n \cos n \phi \left[ I_n(p r) + C_n K_n(p r) \right] \] (36)

Applying the boundary conditions of Eqs (14) and (15) to Eq (36)

\[ \Theta (r = 1, \phi) = f(\phi) = 1 - M + \sum_{n=0}^{\infty} A_n \cos n \phi \left[ I_n(p r) + C_n K_n(p r) \right] \] (37)

and

\[ \frac{\partial \Theta}{\partial r} (r = r_0, \phi) = 0 = \sum_{n=0}^{\infty} \frac{p}{2} A_n \cos n \phi \left[ I_{n-1}(p r_0) + I_{n+1}(p r_0) \right] \]

\[ - C_n \left[ K_{n-1}(p r_0) + K_{n+1}(p r_0) \right] \] (38)
Simultaneous solution of these will yield

\[ A_n = \frac{[K_{n-1}(p_0P_0) + K_{n+1}(p_0P_0)] \left\{ \frac{1}{\pi} \int_0^\pi \{ f(\phi) - 1 + M \} \cos n\phi d\phi \right\}}{[K_{n-1}(p_0P_0) + K_{n+1}(p_0P_0)]I_n(p) + K_n(p)[I_{n-1}(p_0P_0) + I_{n+1}(p_0P_0)]} \]  \hspace{1cm} (39)

and

\[ C_n = \frac{I_{n-1}(p_0P_0) + I_{n+1}(p_0P_0)}{K_{n-1}(p_0P_0) + K_{n+1}(p_0P_0)} \]  \hspace{1cm} (40)

When \( n = 0 \), we get

\[ A_0 = \frac{K_1(p_0P_0) \left\{ \frac{1}{\pi} \int_0^\pi \{ f(\phi) - 1 + M \} d\phi \right\}}{K_1(p_0P_0)I_0(p) + I_1(p_0P_0)K_0(p)} \]  \hspace{1cm} (41)

and

\[ C_0 = \frac{I_1(p_0P_0)}{K_1(p_0P_0)} \]  \hspace{1cm} (42)

Substituting Eqs (39), (40), (41), and (42) into Eq (36), we find the complete solution for \( \theta \) as

\[ \theta = \frac{1}{\pi} \int_0^\pi \{ f(\phi) - 1 + M \} d\phi \left[ \frac{K_1(p_0P_0)I_1(p_0P_0) + I_1(p_0P_0)K_0(p_0P_0)}{K_1(p_0P_0)I_0(p) + I_1(p_0P_0)K_0(p)} \right] \]

\[ \quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \psi(F, n) \cos n\phi \int_0^\pi \{ f(\phi) - 1 + M \} \cos n\phi d\phi \]  \hspace{1cm} (43)

where we have defined

\[ \psi(F, n) = \frac{[K_{n-1}(p_0P_0) + K_{n+1}(p_0P_0)]I_n(p_0P_0) + [I_{n-1}(p_0P_0) + I_{n+1}(p_0P_0)]K_n(p_0P_0)}{[K_{n-1}(p_0P_0) + K_{n+1}(p_0P_0)]I_n(p) + [I_{n-1}(p_0P_0) + I_{n+1}(p_0P_0)]K_n(p)} \]  \hspace{1cm} (44)

Eqs (43) and (44) describe the temperature distribution in the
thin circular fin where the parameters $p$ and $M$ are given by Eqs (21) and (22).

The total heat loss from the fin may be found by applying Fourier's conduction law at the fin root. We define a dimensionless heat loss parameter by

$$\overline{\delta} = \frac{q}{2\pi k_f \delta T_s} = -\frac{1}{\pi} \int_0^\pi \frac{\partial \theta}{\partial \Phi}(\overline{r} = 1, \Phi) d\Phi$$

(45)

Taking the partial derivative of Eq (43) with respect to $\overline{r}$, evaluating at $\overline{r} = 1$, and integrating with respect to $\Phi$, we find from Eq (45) the dimensionless heat loss parameter

$$\overline{\delta} = \Phi\left[ \frac{I_c(p \overline{F}) K_1(p \overline{F}) - K_1(p \overline{F}) I_1(p \overline{F})}{I_c(p \overline{F}) K_0(p \overline{F}) + K_1(p \overline{F}) I_0(p \overline{F})} \right] \left[ \frac{1}{\pi} \int_0^\pi f(\Phi) - 1 + M \Phi d\Phi \right]$$

(46)

Eqs (43) and (46) express the temperature distribution and the amount of heat emitted by a given circular fin with a specified temperature distribution on the inner radius.

Illustrating the application of the solution, two special cases are considered:

Case I. The fin is exposed to still air or air forcibly convected in one direction in the plane of the fin about the tube resulting in a non-uniform temperature distribution at the fin root. This non-uniform fin temperature distribution may well be approximated by

$$f(\Phi) = \cos \Phi \quad 0 < \Phi < \frac{\pi}{2}$$

(47)

The parameter $\Phi$ will be determined approximately by the actual
temperature distribution around the tube. Plots in Figure 2 show the characteristic behavior of $f$.

![Figure 2 The dimensionless temperature distribution at $\bar{T} = 1$ for different $f$.](image)

Substituting Eq (47) into Eqs (43) and (46) and integrating with respect to $\Phi$, we get

$$\theta = 1 - M - (1 - M - \frac{\sin \pi}{9 \pi}) \left[ \frac{K_i(p \bar{P}_0) I_0(p \bar{P}) + I_i(p \bar{P}_0) K_0(p \bar{P})}{K_i(p \bar{P}_0) I_0(p) + I_i(p \bar{P}_0) K_0(p)} \right]$$

$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} \psi(\bar{F}, n) \frac{\cos \phi}{\sqrt{\phi^2 - n^2}}$$

for temperature distribution and

$$\bar{q} = \Phi \left( \frac{\sin \pi}{\bar{T}} - 1 + M \right) \left[ \frac{I_i(p \bar{P}_0) K_i(p) - K_i(p \bar{P}_0) I_i(p)}{I_i(p \bar{P}_0) K_0(p) + K_i(p \bar{P}_0) I_0(p)} \right]$$

for total fin heat loss.
Case II. A simple alternative case is to assume a uniform fin root temperature at one constant value $T_s$ such that

$$f(\phi) = 1$$

which is equivalent to setting $f = 0$ in Eq (47). Therefore, substituting for $f = 0$ into Eqs (48) and (49), we may find

$$\theta = 1 - M \left[ 1 - \frac{K_1(P\bar{r}_0)}{K_1(p\bar{r}_0)} I_0(p) + I_1(p\bar{r}_0) K_0(p) \right]$$

for temperature distribution and

$$\bar{g} = p M \frac{I_1(p\bar{r}_0) I_0(p) - K_1(p\bar{r}_0) I_1(p)}{I_1(p\bar{r}_0) K_0(p) + K_1(p\bar{r}_0) I_0(p)}$$

for total fin heat loss.

C. Sample Problem

For the sake of a sample problem, we shall choose the simple case of an annular fin of constant root temperature in free convection. Consider the following information given:

$$\sigma = 0.1714 \times 10^{-8} \frac{\text{BTU}}{\text{hr-ft}^2 \cdot \text{R}^4}$$

$\Pr = 0.82$

$$\varepsilon = 0.885$$

$k_f = 21 \frac{\text{BTU}}{\text{hr-ft-F}}$

$$T_a = T_R = 700^\circ \text{F}$$

$$R_i = \frac{1}{12} \text{ ft}$$

$$T_s = 1000^\circ \text{R}$$

$D = 0.416 \text{ ft}$
\[ \theta_a = 0.7 \]
\[ \frac{r_o}{r_i} = 2.5 \]
\[ k_a = 0.09 \frac{\text{BTU}}{\text{hr-ft-F}} \]
\[ \delta = \frac{1}{250} \text{ ft} \]
\[ \frac{\rho \beta}{\nu^2} = 1.5(10^6) \frac{1}{\text{ft}^3-\text{F}} \]

We shall suppose that we can consider our circular fin to approximate a vertical flat plate in free convection. The empirical film coefficient correlation given by McAdams [7] for vertical flat plate in laminar free convection is

\[ \frac{h D}{k_a} = 0.59 \left( \frac{\rho \beta}{\nu^2} (T - T_o) D^3 Pr \right)^{1/4} \]  

(53)

We interpret the characteristic dimension D to be the outer diameter of our circular fin. Comparing Eq (53) to Eq (10), we find

\[ G = 0.59 \]
\[ n = m = 1/4 \]

Substituting into Eq (10)

\[ Nu = 0.59 (Gr Pr)^{1/4} = 41.4 \]

From Eqs (11) and (12), we find

\[ B = \frac{2 r_i^2 k_a Nu}{k_f \delta D (1 - \theta_a)^n} = 2 \]

and

\[ C = \frac{8 r_i^2 \delta e T_s^3}{k_f \delta} = 1 \]

Substituting B and C into Eqs (21) and (22), we find

\[ P = \sqrt{\frac{5}{4} \left( B (1 - \theta_a)^n + C \right)} = 1.69 \]
\[ M = \frac{B(1-\Theta_0) \epsilon}{\pi^2} = .233 \]

At \( \bar{r} = 1 \) the fin temperature is \( T_s \) such that \( \Theta = 1 \), and at the tip \( \bar{r} = 2.5 \) with temperature found from Eq (51)

\[
\Theta(\bar{r} = 2.5) = 1 - .233 \left[ 1 - \frac{K_1(4.22) I_0(4.22) + I_1(4.22) K_0(4.22)}{K_1(4.22) I_0(1.69) + I_1(4.22) K_0(1.69)} \right]
\]

\[ = .80 \]

\[ T \left( r = r_0 \right) = 1000 \times .80 = 800^\circ R \]

From Eq (52), the dimensionless heat loss is

\[ \bar{q} = \frac{1}{M} \frac{I_1(4.22) K_0(1.69) - K_1(4.22) I_0(1.69)}{I_1(4.22) K_0(1.69) + K_1(4.22) I_0(1.69)} = .376 \]

\[ q = 2\pi K_f \delta T_s \bar{q} = 198 \frac{BTU}{hr} \]

D. Discussion

The foregoing analysis can be applied to nearly any circular fin on a tube. It takes account of both radiation and convection, forced or free, by applying the appropriate empirical relationship for convection coefficient \( h \).

One mathematical restriction placed upon the analysis was the linearization of the second order partial differential equation. In
the case of free convection and radiation from a circular fin of constant thickness with uniform root temperature, the analysis gave only an approximate solution. The extent of the linearization errors in the temperature distribution and heat loss was determined by comparison of the approximate solution to a numerical computerized solution (termed exact) of the original non-linear differential equation (13). In all cases of comparison, free convection was chosen with \( n = 1/4 \).

Fin temperature distributions were then calculated for two values of the radius ratio, three values of \( \theta_a = \theta_e \), and three values of the convection parameter \( B \) in order to demonstrate the effect of linearizing the convection term. Temperature distributions were also calculated for two values of radiation parameter \( C \) in order to demonstrate the effect of linearizing the radiation term. The approximate and exact comparisons are shown in Figures 3 through 9. A comparison of fin heat losses is shown in Table I. It is interesting to note that for relatively large \( B \), \( C \), and \( \bar{r}_0 \), the dimensionless temperature at the fin tip for the approximate solution asymptotically approaches \( 1 - M \) rather than \( \theta_a = \theta_e \). This fact points out that the linearization tends to give a greater \( p \) and smaller \( M \).

For fixed convection coefficient, we may replace \( \frac{N_u K_a}{D} \) in Eq (11) by \( \bar{H} \) and let \( n = 0 \). By such specification the degree of error appearing in temperature distribution and heat loss will be reduced. For fixed convection coefficient with zero radiation, \( C \) vanishes and the

* The computer program is listed in the Appendix.
<table>
<thead>
<tr>
<th>$\theta_a = \theta_e$</th>
<th>B</th>
<th>C</th>
<th>$R = r_1/r_0$</th>
<th>$\bar{q}$(exact)</th>
<th>$\bar{q}$(approx.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>.5</td>
<td>1</td>
<td>.6</td>
<td>.325</td>
<td>.318</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>.6</td>
<td>.651</td>
<td>.625</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1</td>
<td></td>
<td>1.63</td>
<td>1.47</td>
</tr>
<tr>
<td>.7</td>
<td>.5</td>
<td>1</td>
<td>.6</td>
<td>.215</td>
<td>.212</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>.6</td>
<td>.389</td>
<td>.376</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1</td>
<td></td>
<td>.92</td>
<td>.847</td>
</tr>
<tr>
<td>.9</td>
<td>.5</td>
<td>1</td>
<td>.6</td>
<td>.0836</td>
<td>.082</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>.6</td>
<td>.126</td>
<td>.123</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1</td>
<td></td>
<td>.273</td>
<td>.25</td>
</tr>
<tr>
<td>.5</td>
<td>.5</td>
<td>1</td>
<td>.4</td>
<td>.54</td>
<td>.46</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>.4</td>
<td>.89</td>
<td>.75</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1</td>
<td></td>
<td>1.852</td>
<td>1.49</td>
</tr>
<tr>
<td>.7</td>
<td>.5</td>
<td>1</td>
<td>.4</td>
<td>.363</td>
<td>.322</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>.4</td>
<td>.535</td>
<td>.466</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1</td>
<td></td>
<td>1.0478</td>
<td>.86</td>
</tr>
<tr>
<td>.9</td>
<td>.5</td>
<td>1</td>
<td>.4</td>
<td>.138</td>
<td>.124</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>.4</td>
<td>.18</td>
<td>.16</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1</td>
<td></td>
<td>.3124</td>
<td>.262</td>
</tr>
<tr>
<td>.7</td>
<td>2</td>
<td>1</td>
<td>.4</td>
<td>.461</td>
<td>.423</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>.1</td>
<td>.4</td>
<td>.535</td>
<td>.466</td>
</tr>
</tbody>
</table>

Table I Dimensionless fin heat losses for $n=1/4$. 


solution will be exact. On the other hand, at high altitudes, a space vehicle dissipates heat only by radiation since there is no atmosphere to provide convection so that $B$ vanishes.

Another restriction placed is that the radiant receiver temperature must be a single predictable quantity. It implies that an infinite enclosure exists.

In our analysis we applied an adiabatic boundary condition at the fin tip. The reason for this was because of the assumption that the thickness of the fin was so small that the ratio of the area of the fin tip to other sides was negligible. In addition, the temperature gradient at the fin tip is quite small compared to that at the fin root. These observations allow us to say that the fin tip heat loss is negligible.

In the preceding numerical example, an empirical film coefficient for the vertical flat plate in laminar free convection was used. It was assumed that a round disk with a tube through its middle could be approximated by a vertical flat plate with height equal to the outside diameter of the disk. It is felt that this assumption is within the limits of engineering accuracy and serves to demonstrate the foregoing analysis.
Figure 3 Temperature distribution for variable B for $r_1/r_0 = .6$, $\theta_a = \theta_e = .5$, $C=1$, $n=1/4$. 

$$R = \frac{r_1}{r_0}$$ 

Approximate and exact lines are shown on the graph.
Figure 4 Temperature distribution for variable B
for $r_i/r_o = .6$, $\theta_a = \theta_e = .7$, C=1, n=1/4.
Figure 5 Temperature distribution for variable B for $r_i/r_o = .6$, $\theta_a = \theta_e = .9$, $C = 1$, $n = 1/\delta$. 

$R = r_i/r_o$ 

APPROXIMATE --- - - - - - 

EXACT
Figure 6 Temperature distribution for variable $B$ for $r_1/r_0 = 0.4$, $\theta_a = \theta_e = 0.5$, $C = 1$, $n = 1/4$. 
Figure 7 Temperature distribution for variable B for $r_i/r_o = .4$, $\theta_a = \theta_e = .7$, $C=1$, $n=1/4$. 

$$B = \frac{r_i}{r_o}$$

$R = r_i/r_o$

APPROXIMATE

EXACT
Figure 8 Temperature distribution for variable B for $r_i/r_o = .4$, $\theta_a = \theta_e = .9$, $C=1$, $n=1/4$. 

$R = r_i/r_o$ 

APPROXIMATE 

EXACT
Figure 9 Temperature distribution for variable $C$ for $r_1/r_0 = .4$, $\theta_a = \theta_e = .7$, $B=2$, $n=1/4$. 

$R = r_1/r_0$ 

APPROXIMATE 

EXACT 

$\frac{r - r_i}{r_o - r_i}$
III. ANALYSIS OF HEAT TRANSFER FROM SQUARE FINS
OF CONSTANT THICKNESS

A. Statement of the Problem

A second problem considered is to determine the heat transfer from a square fin of constant thickness on a tube.

The same assumptions applied to the circular fin on a tube will be adopted to the square fin on a tube except assumption 8. In addition, for simplicity, we shall restrict ourselves to consideration of convection with a uniform film coefficient and zero radiation. It will also be assumed that the fin root has a uniform temperature $T_s$. Because of this, the square fin has 45° symmetry such that we may consider only a one-eighth section of the fin bounded by the planes $r = r_1, \phi = 0, \phi = \pi/4$, and $x = b$ as shown in Figure 10.

B. The Solution

Performing a heat balance as we did for the circular fin, we may again derive the governing second order partial differential equation as follows

$$\frac{\delta^2 T}{\delta r^2} + \frac{1}{r} \frac{\delta T}{\delta r} + \frac{\delta^2 T}{\delta \phi^2} = \frac{2 \bar{R}}{K_f \delta} (T - T_a) \tag{54}$$

Because of the restriction of uniform film coefficient and zero radiation, we know that Eq (54) is linear.

To non-dimensionalize, let
Figure 10  Square fin in $r, \phi$ coordinate system.
Substituting these into Eq (54), we get

\[ \frac{\partial^2 \theta}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \theta}{\partial \bar{r}} + \frac{1}{\bar{r}^2} \frac{\partial^2 \theta}{\partial \phi^2} = \bar{p}^2 (\theta - \theta_a) \]  

(56)

The associated boundary conditions will be:

a. At the planes of symmetry

\[ \frac{\partial \theta}{\partial \phi} (\bar{r}, \phi = 0) = \frac{\partial \theta}{\partial \phi} (\bar{r}, \phi = \pi/4) = 0 \]  

(57)

b. At the fin root

\[ \theta (\bar{r} = 1, \phi) = 1 \]  

(58)

c. At the fin tip, assuming negligible heat loss

\[ \frac{\partial \theta}{\partial x} (x = b/f_i, \phi) = 0 \]  

(59)

For simplicity, let \( \bar{\theta} = \theta - \theta_a \) and Eq (56) becomes

\[ \frac{\partial^2 \bar{\theta}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{\theta}}{\partial \bar{r}} + \frac{1}{\bar{r}^2} \frac{\partial^2 \bar{\theta}}{\partial \phi^2} - \bar{p}^2 \bar{\theta} = 0 \]  

(60)

Again, using the method of separation of variables, we assume a product solution

\[ \bar{\theta} = \bar{R}(\bar{r}) \bar{\Phi}(\phi) \]  

(61)

which may reduce Eq (60) to
where either side of the equation, being independent, allows us to assign the constant $k_n^2$. Eq (62) may yield the two ordinary differential equations

$$\frac{1}{R} \left[ R^2 \frac{d^2 R}{d\tau^2} + R \frac{dR}{d\tau} - R^2 \frac{d\Phi}{d\phi^2} \right] = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = k_n^2$$  \hspace{1cm} (62)

Eq (63) is a modified Bessel equation and its solution is

$$R = C_n' I_{k_n}(pR) + D_n' K_{k_n}(pR)$$  \hspace{1cm} (65)

whereas the general solution of Eq (64) is

$$\Phi = A_n' \cos k_n \phi + B_n' \sin k_n \phi$$  \hspace{1cm} (66)

Substituting Eqs (65) and (66) into Eq (61) and using the definitive equation of $\bar{\theta} = \theta - \theta_a$, the general solution for dimensionless temperature distribution becomes

$$\theta = \theta_a + \sum_{n=0}^{\infty} \left[ A_n \cos k_n \phi + B_n \sin k_n \phi \right] \left[ I_{k_n}(pR) + C_n K_{k_n}(pR^2) \right]$$  \hspace{1cm} (67)

where $A_n$, $B_n$, $C_n$, and $k_n$ are arbitrary constants which have absorbed $C_n$.

Applying the adiabatic conditions at $\phi = 0$ and $\pi/4$ in Eq (67)
we set
\[ B_n = 0 \]  
and
\[ \frac{\partial \theta}{\partial \phi} (\bar{r}, \phi = \pi/4) = 0 = \sum_{n=0}^{\infty} A_n K_n \sin n \frac{\pi}{4} [ I_{Kn}(p\bar{r}) + C_n K_n(p\bar{r}) ] \]  
from which we can only conclude that
\[ \sin n \frac{\pi}{4} = 0 \]

hence
\[ K_n = 4n \quad (n = 0, 1, 2, \ldots) \]  

Substituting \( B_n \) and \( k_n \) from Eqs (69) and (71) into Eq (67), we have
\[ \theta = \theta_0 + \sum_{n=0}^{\infty} A_n \cos 4n \phi [ I_{4n}(p\bar{r}) + C_n K_{4n}(p\bar{r}) ] \]  
Using the boundary condition at \( \bar{r} = 1 \) in Eq (72),
\[ \theta(\bar{r}=1, \phi) = 1 = \theta_0 + \sum_{n=0}^{\infty} A_n \cos 4n \phi [ I_{4n}(p) + C_n K_{4n}(p) ] \]  
we find, by the method of Fourier coefficients, that
\[ 1 - \theta_0 = A_0 [ I_0(p) + C_0 K_0(p) ] \]  
and
\[ C_n = - \frac{I_{4n}(p)}{K_{4n}(p)} \]  

(75)

In order to satisfy the remaining adiabatic condition along \( x = b \), we first have to transform the boundary condition in the \( x, y \) system to the \( r, \phi \) system through the fact that \( x = r \cos \phi \) and \( y = r \sin \phi \). The partial derivatives of the composite function \( \Theta(r, \phi) \) are

\[ \frac{\partial \Theta}{\partial r} = \frac{\partial \Theta}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \Theta}{\partial y} \frac{\partial y}{\partial r} = \sin \phi \frac{\partial \Theta}{\partial y} + \cos \phi \frac{\partial \Theta}{\partial x} \]  

(76)

and

\[ \frac{\partial \Theta}{\partial \phi} = \frac{\partial \Theta}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial \Theta}{\partial y} \frac{\partial y}{\partial \phi} = -r \sin \phi \frac{\partial \Theta}{\partial x} + r \cos \phi \frac{\partial \Theta}{\partial y} \]  

(77)

Then, by solving one of these for \( \frac{\partial \Theta}{\partial y} \) and substituting it into the other, we have for \( \frac{\partial \Theta}{\partial x} \)

\[ \frac{\partial \Theta}{\partial x} = \cos \phi \frac{\partial \Theta}{\partial r} - \frac{\sin \phi}{r} \frac{\partial \Theta}{\partial \phi} \]  

(78)

We can determine \( \frac{\partial \Theta}{\partial r} \) and \( \frac{\partial \Theta}{\partial \phi} \) from Eq (72)

\[ \frac{\partial \Theta}{\partial r} = \sum_{n=0}^{\infty} pA_n \cos 4n \phi \left[ \frac{I_{4n-1}(p\phi) + I_{4n+1}(p\phi)}{2} \right. \right. \]

\[ - \frac{I_{4n}(p)}{K_{4n}(p)} \left( \frac{K_{4n-1}(p\phi) + K_{4n+1}(p\phi)}{2} \right) \]  

(79)

and

\[ \frac{\partial \Theta}{\partial \phi} = -\sum_{n=0}^{\infty} 4n A_n \sin 4n \phi \left[ I_{4n}(p\phi) - \frac{I_{4n}(p)}{K_{4n}(p)} \right] \]  

(80)

Applying the adiabatic condition along \( x = b/r_1 \), we have from Eq (78)
Substituting the expressions for \( \frac{\partial \theta}{\partial x} \) and \( \frac{\partial \theta}{\partial \phi} \) from Eqs (79) and (80) into Eq (81), and letting

\[
\bar{p} = p \frac{b}{k_i}
\]

we get

\[
0 = A_0 \left[ J_0(\bar{p} \cos \phi) - C_0 K_0(\bar{p} \cos \phi) \right] + \sum_{n=1}^{\infty} A_n \left[ \gamma(\phi, n) - \frac{J_n(\bar{p})}{K_n(\bar{p})} \zeta(\phi, n) \right]
\]

where we have defined

\[
\gamma(\phi, n) = \cos 4n \phi \left[ I_{4n-1}(\bar{p} \cos \phi) + I_{4n+1}(\bar{p} \cos \phi) - \frac{8n \sin \phi \sin 4n \phi}{\bar{p}} I_{4n}(\bar{p} \cos \phi) \right]
\]

and

\[
\zeta(\phi, n) = \cos 4n \phi \left[ K_{4n-1}(\bar{p} \cos \phi) + K_{4n+1}(\bar{p} \cos \phi) - \frac{8n \sin \phi \sin 4n \phi}{\bar{p}} K_{4n}(\bar{p} \cos \phi) \right]
\]

Up to this point, the constants \( C_0, A_0, A_1, \ldots, A_n \), the total number of which are \( n + 2 \), have not been determined explicitly. We assume that the bound at \( x = b \) is composed of \( n \) points that satisfy the adiabatic condition. To each point \( (\frac{b}{k_i} \cos \phi_i, \phi_i) \) of the bound on \( x = b \), Eq (83) yields a function as
where $i = 0, 1, 2, \ldots, n$. Thus, we shall have $n+1$ equations as follows

\[ A_0 \left( I_1 (\rho \cos^2 \phi) - C_0 K_1 (\rho \cos^2 \phi) \right) + \sum_{n=1}^{\infty} A_n \left[ \eta (\phi, \nu) - \frac{T_{\nu} (\rho)}{K_{\nu} (\rho)} \xi (\phi, \nu) \right] = 0 \]  

(86)

\[ A_0 \left( I_1 (\rho \cos^2 \phi) - C_0 K_1 (\rho \cos^2 \phi) \right) + \sum_{n=1}^{\infty} A_n \left[ \eta (\phi, \nu) - \frac{T_{\nu} (\rho)}{K_{\nu} (\rho)} \xi (\phi, \nu) \right] = 0 \]

\[ A_0 \left( I_1 (\rho \cos^2 \phi) - C_0 K_1 (\rho \cos^2 \phi) \right) + \sum_{n=1}^{\infty} A_n \left[ \eta (\phi, \nu) - \frac{T_{\nu} (\rho)}{K_{\nu} (\rho)} \xi (\phi, \nu) \right] = 0 \]

\[ A_0 \left( I_1 (\rho \cos^2 \phi) - C_0 K_1 (\rho \cos^2 \phi) \right) + \sum_{n=1}^{\infty} A_n \left[ \eta (\phi, \nu) - \frac{T_{\nu} (\rho)}{K_{\nu} (\rho)} \xi (\phi, \nu) \right] = 0 \]  

(87)

Solving Eqs (74) and (87) simultaneously, we are supposed to be able to find the constants. Employing the method of determinants, we find
\[
C_0 = \begin{bmatrix}
I_1(\overline{\Phi}_0) & \eta(\Phi_0,1) + C_1 \xi(\Phi_0,1) & \eta(\Phi_1,1) + C_2 \xi(\Phi_1,2) & \cdots & \eta(\Phi_n,1) + C_n \xi(\Phi_n,n) \\
I_1(\overline{\Phi}_0) & \eta(\Phi_0,1) + C_1 \xi(\Phi_0,1) & \eta(\Phi_1,1) + C_2 \xi(\Phi_1,2) & \cdots & \eta(\Phi_n,1) + C_n \xi(\Phi_n,n) \\
K_1(\overline{\Phi}_0) & \eta(\Phi_0,1) + C_1 \xi(\Phi_0,1) & \eta(\Phi_1,1) + C_2 \xi(\Phi_1,2) & \cdots & \eta(\Phi_n,1) + C_n \xi(\Phi_n,n) \\
K_1(\overline{\Phi}_0) & \eta(\Phi_0,1) + C_1 \xi(\Phi_0,1) & \eta(\Phi_1,1) + C_2 \xi(\Phi_1,2) & \cdots & \eta(\Phi_n,1) + C_n \xi(\Phi_n,n) \\
K_1(\overline{\Phi}_0) & \eta(\Phi_0,1) + C_1 \xi(\Phi_0,1) & \eta(\Phi_1,1) + C_2 \xi(\Phi_1,2) & \cdots & \eta(\Phi_n,1) + C_n \xi(\Phi_n,n) \\
K_1(\overline{\Phi}_0) & \eta(\Phi_0,1) + C_1 \xi(\Phi_0,1) & \eta(\Phi_1,1) + C_2 \xi(\Phi_1,2) & \cdots & \eta(\Phi_n,1) + C_n \xi(\Phi_n,n)
\end{bmatrix}
\]

\[
A_0 = \frac{1 - \Theta_a}{I_0(\phi) + C_0 K_0(\phi)}
\]

and

\[
A_1 = \frac{A_0}{\Delta}
\]

\[
A_1 = \begin{bmatrix}
\mathbf{f}_0 & \eta(\Phi_0,2) + C_2 \xi(\Phi_0,2) & \eta(\Phi_1,3) + C_3 \xi(\Phi_1,3) & \cdots & \eta(\Phi_n,\eta) + C_n \xi(\Phi_n,\eta) \\
\mathbf{f}_1 & \eta(\Phi_1,2) + C_2 \xi(\Phi_1,2) & \eta(\Phi_2,3) + C_3 \xi(\Phi_2,3) & \cdots & \eta(\Phi_n,\eta) + C_n \xi(\Phi_n,\eta) \\
\mathbf{f}_2 & \cdots & \cdots & \cdots & \cdots \\
\mathbf{f}_{n-1} & \eta(\Phi_{n-1},2) + C_2 \xi(\Phi_{n-1},2) & \eta(\Phi_{n-1},3) + C_3 \xi(\Phi_{n-1},3) & \cdots & \eta(\Phi_n,\eta) + C_n \xi(\Phi_n,\eta)
\end{bmatrix}
\]
\[ A_e = \frac{A_0}{\Delta} \]

\[
\begin{align*}
\eta(\Phi_{0,0}) + C_1 \eta(\Phi_{0,1}) & \quad \eta(\Phi_{0,1}) + C_1 \eta(\Phi_{0,2}) & \quad \cdots & \quad \eta(\Phi_{0,n}) + C_1 \eta(\Phi_{0,n-1}) \\
\eta(\Phi_{1,0}) + C_1 \eta(\Phi_{1,1}) & \quad \eta(\Phi_{1,1}) + C_1 \eta(\Phi_{1,2}) & \quad \cdots & \quad \eta(\Phi_{1,n}) + C_1 \eta(\Phi_{1,n-1}) \\
\eta(\Phi_{n-1,0}) + C_1 \eta(\Phi_{n-1,1}) & \quad \eta(\Phi_{n-1,1}) + C_1 \eta(\Phi_{n-1,2}) & \quad \cdots & \quad \eta(\Phi_{n-1,n}) + C_1 \eta(\Phi_{n-1,n-1})
\end{align*}
\]

\[ A_n = \frac{A_0}{\Delta} \]

\[
\begin{align*}
\eta(\Phi_{0,0}) + C_1 \eta(\Phi_{0,1}) & \quad \eta(\Phi_{0,1}) + C_1 \eta(\Phi_{0,2}) & \quad \cdots & \quad \eta(\Phi_{0,n}) + C_1 \eta(\Phi_{0,n-1}) \\
\eta(\Phi_{1,0}) + C_1 \eta(\Phi_{1,1}) & \quad \eta(\Phi_{1,1}) + C_1 \eta(\Phi_{1,2}) & \quad \cdots & \quad \eta(\Phi_{1,n}) + C_1 \eta(\Phi_{1,n-1}) \\
\eta(\Phi_{n-1,0}) + C_1 \eta(\Phi_{n-1,1}) & \quad \eta(\Phi_{n-1,1}) + C_1 \eta(\Phi_{n-1,2}) & \quad \cdots & \quad \eta(\Phi_{n-1,n}) + C_1 \eta(\Phi_{n-1,n-1})
\end{align*}
\]

where we have defined

\[ \Delta = \eta(\Phi_{0,0}) + C_1 \eta(\Phi_{0,1}) \quad \eta(\Phi_{0,1}) + C_1 \eta(\Phi_{0,2}) \quad \cdots \quad \eta(\Phi_{0,n}) + C_1 \eta(\Phi_{0,n-1}) \]

\[ \eta(\Phi_{1,0}) + C_1 \eta(\Phi_{1,1}) \quad \eta(\Phi_{1,1}) + C_1 \eta(\Phi_{1,2}) \quad \cdots \quad \eta(\Phi_{1,n}) + C_1 \eta(\Phi_{1,n-1}) \]

\[ \eta(\Phi_{n-1,0}) + C_1 \eta(\Phi_{n-1,1}) \quad \eta(\Phi_{n-1,1}) + C_1 \eta(\Phi_{n-1,2}) \quad \cdots \quad \eta(\Phi_{n-1,n}) + C_1 \eta(\Phi_{n-1,n-1}) \]
and

\[ f_n = I_1(\bar{p} \cos \phi) - C_0 K_1(\bar{p} \cos \phi) \]  \hspace{1cm} (92)

and \( C_n \) is given by Eq (75).

The complete solution for temperature distribution may be written

\[ \theta = \theta_0 + A_0 \left[ I_0(\bar{p} \bar{r}) + C_0 K_0(\bar{p} \bar{r}) \right] \]

\[ + \sum_{n=1}^{\infty} A_n \cos 4n\phi \left[ I_{4n}(\bar{p} \bar{r}) + C_n K_{4n}(\bar{p} \bar{r}) \right] \]  \hspace{1cm} (93)

where the constants have been determined by Eqs (75), (88), (89), and (90).

The total fin heat loss can be found by applying the Fourier conduction law at the fin root. As in the circular fin case, we define a dimensionless heat loss parameter by

\[ \bar{q} = \frac{q}{2\pi \bar{T}_s \bar{k}_f} = -\frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\partial \theta}{\partial \bar{r}} (\bar{r} = 1, \phi) d\phi \]  \hspace{1cm} (94)

Taking the partial derivative of Eq (93) with respect to \( \bar{r} \), evaluating at \( \bar{r} = 1 \), and integrating with respect to \( \phi \), we find from Eq (94)

\[ \bar{q} = p A_0 \left[ C_0 K_1(p) - I_1(p) \right] \]  \hspace{1cm} (95)

where \( A_0 \) and \( C_0 \) have been determined by Eqs (88) and (89).

The temperature distribution solution as indicated in Eq (93) is exact assuming all terms of the series are evaluated. The infinite series in this particular problem is expected to converge very rapidly. It would allow us to reach a very satisfactory solution with just the
first few terms. We have already stated that the bound at \( x = b \) is considered to be composed of \( n \) number of points at angles of \( \Phi \) that satisfy the adiabatic condition. Selecting a particular number of points on \( x = b \) limits the number of terms in the series solution to the same number. In order to illustrate this, we shall select two points on \( x = b \) for which the adiabatic condition applies and generate a second order solution.

From Eqs (75), (88), (89), and (90), we find

\[
C_0 = \begin{vmatrix}
I_1(\bar{P}\cos^2\Phi_0) & \eta(\Phi_{0,1}) - \frac{I_4(p)}{K_4(p)} \xi(\Phi_{0,1}) \\
I_1(\bar{P}\cos^2\Phi_1) & \eta(\Phi_{1,1}) - \frac{I_4(p)}{K_4(p)} \xi(\Phi_{1,1}) \\
K_1(\bar{P}\cos^2\Phi_0) & \eta(\Phi_{0,1}) - \frac{I_4(p)}{K_4(p)} \xi(\Phi_{0,1}) \\
K_1(\bar{P}\cos^2\Phi_1) & \eta(\Phi_{1,1}) - \frac{I_4(p)}{K_4(p)} \xi(\Phi_{1,1})
\end{vmatrix}

(96)

\[
C_1 = -\frac{I_4(p)}{K_4(p)}
\]

(97)

\[
A_0 = \frac{1 - \theta_a}{I_0(p) + C_0 K_0(p)}
\]

(98)

\[
A_1 = A_0 \frac{p[I_1(\bar{P}\cos^2\Phi_0) - C_0 K_1(\bar{P}\cos^2\Phi_0)]}{\eta(\Phi_{0,1}) - \frac{I_4(p)}{K_4(p)} \xi(\Phi_{0,1})}
\]

(99)
The dimensionless temperature distribution for the fin is

\[ \theta = \theta_0 + A_0 \left[ I_0 (p \tau) + C_0 K_0 (p \tau) \right] + A_1 \cos \phi \left[ I_1 (p \tau) + C_1 K_1 (p \tau) \right] \]  

(100)

and the total heat loss is

\[ \bar{q} = p A_0 \left[ I_1 (p) - C_0 K_1 (p) \right] \]  

(101)

Similarly, for a third order solution, we assume \( \phi_0 \), \( \phi_1 \), and \( \phi_2 \) are the three appropriate values of \( \phi \) where the adiabatic boundary condition is applied on \( x = b \). From Eqs (75), (88), (89), and (90), we find

\[
C_0 = \frac{\begin{vmatrix}
I_1 (p \cos \phi_0) & \gamma (\phi_{0,1}) - \frac{I_4 (p)}{K_4 (p)} \gamma (\phi_{0,1}) & \gamma (\phi_{0,2}) - \frac{I_8 (p)}{K_8 (p)} \gamma (\phi_{0,2}) \\
I_1 (p \cos \phi_1) & \gamma (\phi_{1,1}) - \frac{I_4 (p)}{K_4 (p)} \gamma (\phi_{1,1}) & \gamma (\phi_{1,2}) - \frac{I_8 (p)}{K_8 (p)} \gamma (\phi_{1,2}) \\
I_1 (p \cos \phi_2) & \gamma (\phi_{2,1}) - \frac{I_4 (p)}{K_4 (p)} \gamma (\phi_{2,1}) & \gamma (\phi_{2,2}) - \frac{I_8 (p)}{K_8 (p)} \gamma (\phi_{2,2}) \\
K_1 (p \cos \phi_0) & \gamma (\phi_{0,1}) - \frac{I_4 (p)}{K_4 (p)} \gamma (\phi_{0,1}) & \gamma (\phi_{0,2}) - \frac{I_8 (p)}{K_8 (p)} \gamma (\phi_{0,2}) \\
K_1 (p \cos \phi_1) & \gamma (\phi_{1,1}) - \frac{I_4 (p)}{K_4 (p)} \gamma (\phi_{1,1}) & \gamma (\phi_{1,2}) - \frac{I_8 (p)}{K_8 (p)} \gamma (\phi_{1,2}) \\
K_1 (p \cos \phi_2) & \gamma (\phi_{2,1}) - \frac{I_4 (p)}{K_4 (p)} \gamma (\phi_{2,1}) & \gamma (\phi_{2,2}) - \frac{I_8 (p)}{K_8 (p)} \gamma (\phi_{2,2}) \\
\end{vmatrix}}{\begin{vmatrix}
I_0 (p) & I_0 (p) & I_0 (p) \\
I_0 (p) & I_0 (p) & I_0 (p) \\
I_0 (p) & I_0 (p) & I_0 (p) \\
K_0 (p) & K_0 (p) & K_0 (p) \\
K_0 (p) & K_0 (p) & K_0 (p) \\
K_0 (p) & K_0 (p) & K_0 (p) \\
\end{vmatrix}} \]  

(102)

\[ C_1 = - \frac{I_4 (p)}{K_4 (p)} \]  

\[ C_2 = - \frac{I_8 (p)}{K_8 (p)} \]  

(103)
\[ A_o = \frac{1 - \theta_o}{I_o(p) + C_o K_o(p)} \] (104)

\[ A_1 = A_0 \frac{I_1(\Phi \cos^2 \Phi_o) - C_0 K_1(\Phi \cos^2 \Phi_o) \gamma(\Phi_{o,1}) + C_2 \gamma(\Phi_{o,2})}{\gamma(\Phi_{o,1}) + C_1 \gamma(\Phi_{o,1}) \gamma(\Phi_{o,2}) + C_2 \gamma(\Phi_{o,2})} \] (105)

\[ A_2 = A_0 \frac{\gamma(\Phi_{o,1}) + C_1 \gamma(\Phi_{o,1})}{\gamma(\Phi_{o,1}) + C_1 \gamma(\Phi_{o,1})} \frac{I_1(\Phi \cos^2 \Phi_o) - C_0 K_1(\Phi \cos^2 \Phi_o)}{\gamma(\Phi_{o,1}) + C_1 \gamma(\Phi_{o,1}) \gamma(\Phi_{o,2}) + C_2 \gamma(\Phi_{o,2})} \] (106)

The temperature distribution for the fin is

\[ \theta = \theta_o + A_0 \left[ I_o(p \tilde{F}) + C_o K_o(p \tilde{F}) \right] + A_1 \cos 4 \Phi \left[ I_4(p \tilde{F}) + C_4 K_4(p \tilde{F}) \right] + A_2 \cos 8 \Phi \left[ I_8(p \tilde{F}) + C_8 K_8(p \tilde{F}) \right] \] (107)
and the total loss is again

\[ \bar{g} = p A_0 \left[ C_0 K_1(p) - I_1(p) \right] \quad (108) \]

C. Sample Problem

For a sample problem, we shall work out the second order solution. \( \phi_0 = 20^\circ \) and \( \phi_1 = 41^\circ \) are to be chosen for the adiabatic condition on \( x = b \). Consider the following information given:

- \( h = 9 \ \text{BTU/hr-ft-}^\circ\text{F} \)
- \( \delta = \frac{1}{240} \ \text{ft} \)
- \( k_f = 30 \ \text{BTU/hr-ft-}^\circ\text{F} \)
- \( r_1 = \frac{1}{12} \ \text{ft} \)
- \( T_s = 715^\circ \text{R} \)
- \( b/r_1 = 2 \)
- \( T_a = 5000^\circ \text{R} \)

From these we can find

\[ p = 1 \quad \bar{p} = p \frac{b}{r_1} = 2 \quad \theta_a = .7 \]

and therefore

\[ \bar{p} \cos^{-1} \phi_0 = 2.13 \quad \bar{p} \cos^{-1} \phi_1 = 2.65 \]

From Eq (84)
\[ \eta(\Phi = 20^\circ, 1) = \cos 4\Phi_0 \left[ I_3(2.13) + I_5(2.13) \right] - \frac{85 \sin \Phi_0 \sin 4\Phi_0}{2} I_4(2.13) \]

\[ \eta(\Phi = 20^\circ, 1) = 1.37 \]

and

\[ \eta(\Phi = 41^\circ, 1) = -0.42 \]

From Eq (85)

\[ \xi(\Phi = 20^\circ, 1) = \cos 4\Phi_0 \left[ K_3(2.13) + K_5(2.13) \right] - \frac{85 \sin \Phi_0 \sin 4\Phi_0}{2} K_4(2.13) \]

\[ \xi(\Phi = 20^\circ, 1) = 0.925 \]

and

\[ \xi(\Phi = 41^\circ, 1) = -1.78 \]

From Eq (97)

\[ C_1 = -\frac{I_4(\nu)}{K_4(\nu)} = -6.2 \times 10^{-5} \]

Substituting these into Eqs (96), (98), and (99), we find

\[ C_0 = \begin{vmatrix} I_1(2.13) & \eta(\Phi_0, 1) + C_1 \xi(\Phi_0, 1) \\ I_1(2.65) & \eta(\Phi_0, 1) + C_1 \xi(\Phi_0, 1) \\ K_1(2.13) & \eta(\Phi_0, 1) + C_1 \xi(\Phi_0, 1) \\ K_1(2.65) & \eta(\Phi_0, 1) + C_1 \xi(\Phi_0, 1) \end{vmatrix} = 29.5 \]
$$A_0 = \frac{1 - \Theta_a}{I_0(\xi) + 29.5 \, K_0(\xi)} = 0.0327$$

and

$$A_1 = A_0 \frac{p[I_1(2.13) - C_0 K_1(2.13)]}{\gamma(\phi_1) + C_1 \xi(\phi_1)} = 0.115$$

The dimensionless temperature distribution of the square fin will be from Eq (100)

$$\theta = 0.7 + 0.0327 \left[ I_0(\xi) + 29.5 \, K_0(\xi) \right] + 0.115 \cos 4\phi \left[ I_4(\xi) - 6.2 \times 10^{-5} K_4(\xi) \right]$$

The temperature at the corner ($\xi = 2 \cos^{-1} 45^\circ$, $\phi = 45^\circ$) is found to be

$$\theta = 0.7 + 0.0327 \left[ I_0(2.84) + 29.5 \, K_0(2.84) \right] + 0.115 \cos 4\phi \left[ I_4(2.84) - 6.25 \times 10^{-5} K_4(2.84) \right] = 0.835$$

$$T = 0.835 T_s = 597^\circ R$$

The total fin heat loss will be from Eq (101)
A series solution was found for the temperature distribution and heat loss from a square fin of constant thickness with uniform root temperature.

It is clear that the second or third order solution for temperature distribution means that we are using only two or three terms respectively of the exact series solution. If we consider more points on \( x = b \), then we are not only approaching the true adiabatic condition along \( x = b \), but also retaining a larger number of terms in the series solution. It turns out that the zero order solution of Eq (93) provides the temperature distribution for an equivalent circular fin on a tube with uniform root temperature, the outer radius of which is equal to the distance from tube axis to the selected point on \( x = b \). The higher order terms in Eq (93) serve as corrections to the circular fin to construct the true geometry of the square fin.

It can be shown that the third order solution is an excellent approximation to the full series solution by comparing a second order solution with the corresponding third order solution for a typical square fin as represented in the sample problem. The temperature
distributions for each of these second and third order solutions are plotted in Figure 11. As can be seen, the temperatures differ only slightly. The corresponding heat losses are almost identical in value. The magnitude is 199 BTU/hr.

One question here is how to select the values of \( \phi \) in order to find a good solution. About the best we can do is to see how closely the solution satisfies the adiabatic condition all along the boundary at \( x = b \). As a means of doing this, we compare the temperature gradients at points on \( x = b \) with those at corresponding angles \( \phi \) on \( r = r_1 \) where the gradients are highest. The best choice of \( \phi \)'s will give an evenly distributed percentage of error as shown in Figure 12.

One may pose the question, "what is the effect of changing the fin size upon the selection of \( \phi \)'s?" Figure 13 shows that, for two different fin sizes, the best selection of \( \phi \)'s are very nearly the same. This is indicated by the fact that the most even error distribution for the two cases exhibited the crossings of the abscissa at very nearly the same angle \( \phi \). Based on the sample work, it is suggested that \( \phi_1 = 20^\circ \), \( \phi_2 = 41^\circ \) be used for the second order solution while \( \phi_3 = 18^\circ \), \( \phi_4 = 30^\circ \), \( \phi_5 = 43^\circ \) be used for the third order solution.

It is understood that \( 45^\circ \) symmetry of a square fin would not ordinarily be met with great accuracy in practice; neither would the uniform root temperature. However, it is felt that the solution for this particular case should approximate the result that would be found in practice with a reasonable degree of accuracy.
3rd order solution with $\phi = 18^\circ, 30^\circ, 43^\circ$

2nd order solution with $\phi = 20^\circ, 41^\circ$

Figure 11 Temperature distribution in square fin with $p = 1$, $\theta_a = .7$, and $b/r_1 = 2$. 
Figure 12 Percent error of adiabatic condition at $x = b$ in second and third order solution.

Figure 13 Percent error of adiabatic condition at $x = b$ in second order solution for different $b/r_1$. 
One more point of discussion should take up the fact that radiation and temperature dependent film coefficient were omitted from consideration in the present problem but were accounted for in the circular fin problem. It should be realized, however, that these factors can be accounted for in the present square fin problem if we redefine certain of the parameters. In Eq (55) the parameter $p$ should be redefined as in Eq (21) and $\theta_a$ should be replaced by $(1 - M)$ where $M$ is given by Eq (22). These changes would then yield a linearized governing differential equation for the square fin as given by Eq (24). Substituting these new definitions into Eqs (93) and (95) would yield an approximate solution for radiation and convection from the square fin with the film coefficient as a variable function of local fin temperature. The resulting solution would be in error to about the same degree as the comparable circular fin solution.
IV. CONCLUSIONS AND RECOMMENDATIONS

Chapters II and III of the thesis presented the solutions for the temperature distribution and heat loss of a circular fin and a square fin respectively. In the circular fin case, the analysis accounted for both radiation and variable film coefficient convection from the surface of the fin. It was shown that the resulting governing equation was non-linear. By employing the binominal theory, linearization was effected and an approximate solution was derived. The result, as demonstrated by examples, exhibited acceptable accuracy for most practical examples.

In deriving a solution for the square fin, only a fixed film coefficient was accounted for in order to maintain linearized governing differential equation. The resulting temperature distribution solution was an infinite series which proved to be rapidly convergent, yielding a solution very nearly exact when taken to third order. Upon comparing certain parameters in the square fin problem to the comparable ones in the circular fin problem, as mentioned in the discussion section of chapter III, it was realized that radiation and variable film coefficient convection could also be accounted for in an approximate manner in the square fin problem.

It is thus concluded that the solutions provided in the thesis are quite good and therefore should be of benefit to heat transfer designers working in the field of finned-tube exchangers. It would be recommended that analysis of the nature provided in this thesis be
given further consideration in attempting to provide solutions for other fin configurations. One such example might be the consideration of the whole square fin, rather than a one-eighth symmetry segment, in which case it might be possible to allow for a variable fin root temperature as was done for the circular fin. It has been pointed out that a circular fin could be considered as only one special case of the square fin. Similarly, one might attempt to find a solution for a rectangular fin where the square fin would be considered a special case of it.
REFERENCES


APPENDIX

The computer program used for the exact solution of Eq (13) in the circular fin case utilized the Runge-Kutta method [8]. The program stated in Fortran language with \( n = \frac{1}{4} \) and \( \frac{\partial^2 \phi}{\partial r^2} = 0 \) is listed below:

1 FORMAT (45H RUNGE KUTTA METHOD FOR HEAT TRANSFER PROBLEM)

2 FORMAT (15H YUNSENG HUANG)

3 FORMAT (7 F10.4)

10 READ 3, H, D, AL, B, C, E, F

E=1.

13 PUNCH 3, H, D, AL, B, C, E, F

P=F*H
Q=(-F/D+B*((E-AL)**1.25)+C*(E**4.-AL**4.)/4.)*H
R=(F+Q/2.)*H
G=D*H/2.
O=E+P/2.
S=(-R/(G*H)+B*((0-AL)**1.25)+C*(0**4.-AL**4.)/4.)*H
T=(F+S/2.)*H
A=E+R/2.
U=(-T/(G*H)+B*((A-AL)**1.25)+C*(A**4.-AL**4.)/4.)*H
V=(F+T)*H
AO=E+U-AL
W=(-(F+T)/(D+H)+B*(AO**1.25)+C*((E U)**4.-AL**4.)/4.)*H
X=(P+2.*R+2.*T+V)/6.
Y=(Q+2.*S+2.*U+W)/6.
E = E + X
F = F + Y
20 D = D + H
GO TO 13
41 GO TO 10
END