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Turbulent Velocity and Temperature in the Mixing Region of Two Parallel Airstreams

Olin Thomas Sessions
TURBULENT VELOCITY AND TEMPERATURE
IN THE MIXING REGION OF TWO
PARALLEL AIRSTREAMS

BY

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TURBULENT VELOCITY AND TEMPERATURE
IN THE MIXING REGION OF TWO
PARALLEL AIRSTREAMS

This thesis is approved as a creditable and
independent investigation by a candidate for the degree,
Master of Science, and is acceptable as meeting the thesis
requirements for this degree, but without implying that
the conclusions reached by the candidate are necessarily
the conclusions of the major department.

Thesis Adviser

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OTS
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NOMENCLATURE

A  - hot wire sensitivity to velocity fluctuations
a  - momentum mixing region half thickness, to the left of the center line
at - thermal mixing region half thickness, to the left
B  - hot wire sensitivity to temperature fluctuations
b  - momentum mixing region half thickness, to the right
bt - thermal mixing region half thickness, to the right
C  - instantaneous air cooling velocity
C_p - specific heat of air at constant pressure
D  - hot wire diameter
E  - instantaneous hot wire voltage
E_f - amplifier output, resulting from fluctuations of the air stream
E_s - amplifier output, resulting from an imposed square wave
H  - overall heat transfer coefficient of hot wire
I  - electrical current through hot wire
i  - square wave current
K  - thermal conductivity of air
K_l - amplifier constant
K_v - voltmeter constant
L  - mixing length, according to Prandtl
l_w - mixing length, according to Taylor
M  - root mean square value of anemometer voltage output
m - empirical constant, correcting for effects of wire of finite length

N - constant, defined as \( \frac{R_o + d}{T} \)

\( \text{Nu} \) - Nusselt number, \( \frac{\alpha x}{K} \)

P - pressure

Pr - Prandtl number, \( \frac{\nu C_p}{K} \)

\( \text{Pr}_t \) - turbulent Prandtl number, \( \frac{\varepsilon_m C_p}{\varepsilon_\theta} \)

q - heat transfer per unit area

R - instantaneous electrical resistance of hot wire

\( R_o \) - instantaneous resistance of wire at air temperature, with current off

Re - Reynolds' number, \( \frac{\rho u x}{\nu} \)

r - resistivity of wire

S - constant, defined as \( 0.3 \left( \frac{\pi D}{2} \right)^2 \left( \frac{K_*}{\alpha r} \right) \)

T - temperature

\( T_w \) - instantaneous wire temperature

t - time

U - free stream velocity

u - velocity component in the x-direction

v - velocity component in the y-direction

w - velocity component in the z-direction

x - longitudinal coordinate in the direction of flow, \( x=0 \) corresponds to the location where streams start to mix

y - lateral coordinate in the transverse direction of flow, \( y=0 \) corresponds to the splitter plate location

z - lateral coordinate perpendicular to the x-y plane
\( \alpha \) - temperature coefficient of resistance for a hot wire

\( \beta \) - angle between the free stream direction and the hot wire

\( \epsilon_m \) - apparent viscosity

\( \epsilon_\varphi \) - apparent conductivity

\( \gamma \) - intermittency factor

\( \lambda \) - amplifier time constant

\( \psi \) - wire time constant

\( \rho \) - density of air

\( \sigma \) - resistance ratio, \( R/R_0 \)

\( \tau \) - shear stress

\( \nu \) - viscosity coefficient of air

\( \omega \) - vorticity, \( \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \)

\( \chi \) - constant exchange coefficient, according to Prandtl

\((\ )^*\) - arbitrary reference state

\((\ )\) - mean time average of a term

\((\ )'\) - difference between instantaneous and mean time average values of a term

\((\ )_1\) - denotes output of wire one

\((\ )_2\) - output of wire two

\((\ )_1+2\) - sum of outputs of wires one and two

\((\ )_{1-2}\) - difference of outputs of wires one and two

\((\ )_{\max}\) - maximum value of a term

\((\ )_{\min}\) - minimum value of a term

\((\ )_{\text{ave}}\) - average value of a term
CHAPTER I

INTRODUCTION

Two basic types of fluid motion are known, laminar and turbulent flow. Laminar flow is characterized by essentially uniform flow of fluid particles and obedience to Newton's Viscosity Law, \( \tau = \mu \frac{dU}{dy} \). Many fluids have been found to have dynamic viscosities (\( \mu \)) which are essentially constant and are considered to be properties of the fluid. The viscosities of gases can even be derived by the kinetic theory of gases and are found to be functions of the mean free path of individual molecules. Turbulent motion, on the other hand, can be described as a flow made up of macroscopic lumps of fluid which fluctuate in a random manner.

Most fluid flows occurring in practical situations are of a turbulent nature. The fluctuations or eddying movements can produce substantial changes in the nature of the fluid behavior (e.g., greatly increased shear stresses and heat transfer) and consequently are of much importance to the engineer dealing with them. Unfortunately, turbulent flows are not readily amenable to exact mathematical analyses.

Turbulent flows can be categorized under the following headings: (1) homogenous isotropic flow
The development of the hot-wire anemometer has permitted quantitative studies of the nature of turbulence, and forms the basis of the present experimental work. Corrsin, 1* in 1947, developed equations enabling the measurement of both velocity and temperature fluctuations with a hot-wire anemometer, and his equations will be extended to the present case.

The present study was conducted on a mixing region formed by the joining of two streams flowing parallel in a duct initially separated by a splitter plate (see following chapter). This type of flow is particularly conducive to the study of turbulence because of the absence of laminar effects due to walls. By studying two streams at different temperatures, it was hoped that some additional light could be shed on the relationship of momentum transfer to the transfer of heat in turbulent flow.

Finally, the equations 2 on which the study is based should be noted. These equations are the Navier-Stokes equations of motion and the energy equation.

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*Cited literature will be referenced to the bibliography.
which have been derived assuming a continuum. The general nature of these equations permits a mathematical analysis of almost any type of Newtonian fluid motion provided that sufficient restrictions are applied in order to reduce the equations to integrable forms and sufficient boundary conditions can be applied. Some researchers have questioned the applicability of these equations to turbulence, suggesting that the scale of turbulence may approach the molecular mean free path and hence violate the assumption of a continuum. Taylor, however, seems to have clearly established that the scale of turbulence is considerably larger than the mean free path. As a result, the Navier-Stokes equations are almost universally used in dealing with turbulence.
CHAPTER II

RESEARCH FACILITIES

The experimental investigation was carried out in a wind tunnel designed specifically for the study of a turbulent mixing region. The tunnel is 10 inches square at the test section and the air flow is divided into two channels, by a splitter plate, up to the test section where the two flows are allowed to merge. A schematic of the test section is shown in Figure 1.* The facility has been more thoroughly described by Iverson, Goel and Kascoutas.

Kascoutas used screens mounted 15 inches upstream of the end of the splitter plate in an attempt to produce a thickened boundary layer. Because of the excessive turbulence he encountered, it was decided to remove the screens, in the hope that the flow characteristics of the tunnel could be improved.

The facility is instrumented with a hot-wire anemometer and related electronic equipment. The hot-wire anemometer is capable of measuring average velocities and temperatures. It is also capable of producing voltage fluctuations which result from the fluctuation of

*Figures 1 and 2 will be included in the text, all following figures will be shown in Appendix A.
Figure 1. Wind Tunnel Test Section
velocity and temperature in the air stream. These fluctuating voltages may be measured by the use of a random signal voltmeter, and, with the proper equations, a relationship of the voltage fluctuations to the turbulent flow properties can be established. A sum and difference control unit, in conjunction with the proper sensing device, permits the simultaneous operation of two hot wires in an X-array allowing experimental determination of the transverse component of velocity. The correlation of the longitudinal and transverse components of velocity can also be determined, as well as the correlation of the velocity components with temperature fluctuations.
CHAPTER III

EQUATIONS OF MOTION AND ENERGY

The continuity equation based upon conservation of mass, the Navier-Stokes equations of motion based upon Newton's second law and the energy equation based upon the first law of thermodynamics form the basis for solutions to fluid dynamics problems. The following are the three equations in vector form,

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0 \]  \hspace{1cm} (3-1)

\[ \rho \frac{\partial \vec{v}}{\partial t} = -\frac{1}{3} \rho \vec{v} \left( \text{div} \vec{v} \right) + \rho \nabla^2 \vec{v} = - \nabla P \]  \hspace{1cm} (3-2)

\[ \rho C_p \frac{\partial T}{\partial t} = \frac{\partial P}{\partial t} + \nabla \cdot K \nabla T \]  \hspace{1cm} (3-3)

where viscous dissipation has been neglected. These equations obviously cannot be solved in their present form. It will therefore be necessary to make some restrictive assumptions.

The first and most general of these assumptions is that of two-dimensionality.\textsuperscript{6} This restriction has the effect of eliminating all \( w \) and \( \partial / \partial z \) terms.

An additional assumption applicable to turbulent mixing zones is the assumption that the pressure is constant throughout the flow field. This was observed to be true over the 15 inch longitudinal section comprising the mixing region in the present problem. Thus \( \frac{\partial P}{\partial t} \) is equal
to zero.

These assumptions reduce (3-1), (3-2) and (3-3) to the following:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \] \hspace{1cm} (3-4)

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{1}{2} \mu \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \] \hspace{1cm} (3-5)

\[ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \frac{1}{2} \mu \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]

\[ \rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) \] \hspace{1cm} (3-6)

A third and much more restrictive assumption concerns the density. The present case deals with low velocities \((M<0.05)\) and a relatively small temperature difference \((\Delta T<40^\circ F)\). Thus the density will not vary greatly, resulting in the derivatives of density in (3-4) being of smaller orders of magnitude than the corresponding derivatives of velocity; therefore the continuity equation reduces to \(\text{div} \mathbf{V} = 0\) which implies that the flow behaves essentially as though it were incompressible. Fluid properties such as viscosity and thermal conductivity which vary little with temperature may also be considered as constant properties. This approach has been used by other authors for similar cases.7,8
Because there is no solid boundary in a mixing zone, velocity gradients with respect to $y$ are much less severe than those in boundary layer flow. This has been demonstrated by several researchers.\textsuperscript{9,10} The combination of this fact and the small value of viscosity of air can be used to produce another assumption,\textsuperscript{6} that is, the remaining terms on the right of (3-5) may be completely removed.

Now, examining the remaining terms of (3-6), it may be noted that the left hand side of the equation denotes the convective transfer of heat while the right hand side denotes conduction. Since the thermal conductivity ($K$) is very small for air, it is to be expected that the terms on the left greatly outweigh the terms on the right. Thus the terms on the right may be eliminated.

Rewriting (3-4), (3-5) and (3-6) in light of the preceding assumptions, the following equations are found:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3-7)$$

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = 0 \quad (3-8)$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = 0 \quad (3-9)$$

Each instantaneous dependent variable occurring in (3-7), (3-8) and (3-9) may be represented as the summation of mean time average components and fluctuating
components. For instance, the instantaneous value of a property $B$ may be represented as

$$B = \overline{B} + B'$$  \hspace{1cm} (3-10)

Now the instantaneous terms of (3-7), (3-8) and (3-9) may be written in the form of (3-10). Substituting these terms, taking a mean time average, and noting that there is no reason to expect the mean time average of $u$, $v$, or $T$ to vary with time, the following equations result:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$  \hspace{1cm} (3-11)

$$\rho (\overline{u} \frac{\partial \overline{u}}{\partial x} + \overline{v} \frac{\partial \overline{u}}{\partial y}) = -\rho \left[ \frac{\partial}{\partial x} (\overline{u^2}) + \frac{\partial}{\partial y} (\overline{u'v'}) \right] \hspace{1cm} (3-12)$$

$$\rho (\overline{u} \frac{\partial \overline{v}}{\partial x} + \overline{v} \frac{\partial \overline{v}}{\partial y}) = -\rho \left[ \frac{\partial}{\partial x} (\overline{u'v'}) + \frac{\partial}{\partial y} (\overline{v'^2}) \right] \hspace{1cm} (3-13)$$

Since these equations are still too complex to permit solution, further simplifications are necessary. It may be anticipated that reductions can be made in the turbulence terms. These reductions, however, will require additional experimental information.
CHAPTER IV

REVIEW OF LITERATURE

A considerable body of work has been done on the problem of turbulence. Before attempting a review of this work it appears reasonable to define the present problem more specifically, making it possible to limit the review to relevant material.

The problem to be studied will be that of the mixing of two parallel streams of air, flowing at the same velocity but at different temperatures. The initial boundary layer will be considered, thus velocity distributions similar to those of a plane wake may be anticipated. Figure 2 shows the expected profiles of velocity and temperature. As shown in the figure, no geometric similarity of the two profiles will be possible. The case will also be limited to that of low velocities and a small temperature difference, as stated in Chapter III.

Now it seems logical to study the work of previous researchers, in the hope that some insight relevant to the present study may be gained. The review will be divided into two sections, a review of the various phenomenological theories which have arisen and a review of the experimental work.

A. Theories of Turbulence

Because of the complicated nature of turbulence
Figure 2. Anticipated Velocity and Temperature Profiles at \( x = 0 \)
it does not appear that a complete understanding of its mechanism may be attained. Since the values of prime interest are the mean velocities and temperatures, a number of semi-empirical theories have been postulated in an attempt to build a mathematical basis for the investigation of the mean properties of turbulent flows. It should be noted, because of the oversimplifications introduced in these theories, that even if some of the simplifications are proven false, the theories may still be of use if they permit the calculation of mean values in reasonable agreement with experiment.

Boussinesq, working with Reynold's idea of apparent shear stress, developed the first of these theories. He proposed a relationship between the apparent shear and the velocity gradient, in analogy with the equation for laminar shear.

\[ \tau_{app} = -\rho \overline{(u'v')} = \rho \epsilon_m \frac{d\overline{u}}{dy} \]  

A similar relationship has been proposed for the transfer of heat, analogous to Fourier's law for laminar flow

\[ q_{app} = \rho c_p \overline{(v'T')} = -\rho \epsilon_q \frac{dT}{dy} \]  

Unfortunately, the apparent viscosity (\(\epsilon_m\)) and apparent conductivity (\(\epsilon_q\)) have turned out to be complicated functions of the local flow conditions, and thus of little use in the solution of the equations of motion and energy. They may, however, be of value in relating the mechanisms
of heat transfer and momentum transfer.

Prandtl\(^2\) developed the first workable phenomenological theory by making an assumption about the mechanism of momentum transfer. He assumed that turbulence could be characterized by the coalescing of fluid into lumps which move as one body. These fluid lumps jump transversely from streamline to streamline, resulting in a transfer of momentum. By assuming a characteristic length over which these lumps traverse, he was able to obtain

\[ |\bar{u}'| = \lambda \frac{d\bar{u}}{dy} \quad (4-3) \]

where \(\lambda\) is the characteristic length and has come to be known as Prandtl's mixing length. Going further, Prandtl attempted to explain the origin of the lateral fluctuations. By proposing a collision of two fluid lumps converging on a streamline from opposite directions, one with a higher velocity than the median streamline and the other with a lower velocity, he was able to say that a transverse fluctuation on the same order of magnitude as the longitudinal fluctuation arose when these lumps diverged after the collision. From this argument he wrote

\[ |\bar{v}'| = \text{const} \times |\bar{u}'| \quad (4-4) \]

Therefore

\[ |\bar{u}'| \cdot |\bar{v}'| = \text{const} \times \lambda^2 \times \left( \frac{d\bar{u}}{dy} \right)^2 \quad (4-5) \]

It may be reasoned that in the region of a positive mean velocity gradient the instantaneous fluctuations will have
opposite signs. Using this convention the mean time average product of \(u'\) and \(v'\) may be related to the product of their absolute magnitudes according to

\[
\langle u'v' \rangle = -t \langle u' \rangle \langle v' \rangle
\]  
(4-6)

where \(t\) represents some fractional value. There results a relationship for \(T_{APP}\),

\[
T_{APP} = -\rho \langle u'v' \rangle = \rho L^2 \left( \frac{dU}{dy} \right)^2
\]  
(4-7)

where \(L^2\) has now absorbed both \(t\) and the previously written constant.

The apparent heat transfer may also be found by similar arguments. It must be pointed out, however, that the arguments used assume that heat is transferred by the same mechanism, an assumption which may not be valid. The primary assumption is that

\[
|T'| = \gamma \frac{dT}{dY}
\]  
(4-8)

where \(\gamma\) is the same mixing length that occurs in (4-3). This results in an expression for \(q_{APP}\) similar to (4-7).

\[
q_{APP} = \rho C_p \langle T' \rangle = \rho C_p L^2 \frac{dU}{dy} \cdot \frac{dT}{dY}
\]  
(4-9)

It must be emphasized that this approach assumes the mechanisms of momentum and heat transfer to be the same, and consequently will result in equal velocity and temperature distributions.

Using the conclusions of Prandtl's theory, it is now possible to apply an order of magnitude analysis to (3-12) and (3-13) in an attempt to determine whether the
contribution of some of the terms may be negligible.

By noting that the mixing region is much longer than it is wide, it is possible to assign orders of magnitude to some of the terms in (3-12). Assume that a scale of measure may be assigned such that $U_{max}$ (free stream velocity) and $x$ (downstream dimension) are of the same order of magnitude, and define this to be of order unity, $O(1)$. Now since the width of the mixing region is much less than the length, define $\gamma$ to be $O(\delta)$ where $O(1) \gg O(\delta)$. Substituting the relative orders of magnitude into the continuity equation (3-11),

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\frac{O(1)}{O(1)} + \frac{O(\bar{u})}{O(\delta)} = 0$$

it becomes obvious that $\bar{v}$ is $O(\delta)$.

Utilizing these orders of magnitude, Kascoutas was able to reduce (3-12) to the form

$$\rho \left( \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) = -\rho \frac{\partial}{\partial y} (\bar{u}' \bar{v}')$$

subject to experimental confirmation that $\frac{\partial}{\partial x} (\bar{u}' \bar{v}') < \frac{\partial}{\partial y} (\bar{u}' \bar{v}')$ and that $\frac{\partial}{\partial x} (\bar{u})^2 << \frac{\partial}{\partial y} (\bar{u}' \bar{v}')$. It should be noted that Kascoutas found the second equation of (3-13) to be entirely $O(\delta)$, and thus decided it was negligible compared to the first.

If in a similar manner the mean temperature ($\bar{T}$) is assigned to be $O(1)$, a similar reduction may be applied to
the energy equation. Substituting these values in (3-13) produces

\[ \rho C_p \left( \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} \right) = -\rho C_p \left[ \frac{3}{\partial x} (\bar{u}'\bar{T}') + \frac{3}{\partial y} (\bar{v}'\bar{T}') \right] \]

\[ \rho C_p \left( o(u) \frac{\partial (u)}{\partial u} + o(\delta) \frac{\partial (u)}{\partial \delta} \right) = -\rho C_p \left[ \frac{\partial (u'\bar{T}')}{\partial (u)} + \frac{\partial (v'\bar{T}')}{\partial (\delta)} \right] \]

Now it is necessary to make assumptions regarding the order of magnitude of \((u'\bar{T}')\) and \((v'\bar{T}')\). Utilizing Prandtl's assumption that \(\bar{u}'\) is of the same order of magnitude as \(\bar{v}'\) it can be reasoned by the same arguments that produced (4-6) that \((u'\bar{T}')\) is of the same order of magnitude as \((v'\bar{T}')\),

\[ O(u'T') = O(v'T') \]

This produces the supposition that \(\frac{3}{\partial x} (u'\bar{T}') \ll \frac{3}{\partial y} (v'\bar{T}')\) and thus the term \(\frac{3}{\partial x} (u'\bar{T}')\) may be dropped from (4-12), resulting in

\[ \rho C_p \left( \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} \right) = -\rho C_p \frac{3}{\partial y} (v'\bar{T}') \]

(4-13)

One of the more pertinent facts about this analysis is that no assumptions have been made about the magnitude of \(\bar{T}'\). Thus it may be possible that the term containing \(\bar{T}'\) is actually small compared to the other terms of (4-13), though this is not the anticipated result.

Taylor,\(^3\) in comparing the transfer of momentum with that of heat in free turbulence, reasoned that their mechanisms would be fundamentally different. Rather than corresponding directly with momentum transfer, he thought the transfer of heat would correspond to the transfer of
vorticity. By defining a vorticity term
\[ \omega = \frac{1}{2} \left( \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} \right) \]  
and utilizing Euler's equation of motion, he was able to develop an equation for the apparent shear.
\[ \tau_{APP} = \frac{1}{2} \rho l_w^2 \left( \frac{dU}{dY} \right)^2 \]  
This expression is essentially the same as (4-7) with \( l_w \) identical to \( \sqrt{2} l \). However, since he predicted that heat would be transferred in correspondence with vorticity rather than momentum, a slightly different expression for the apparent heat transfer resulted.
\[ q_{APP} = \rho C_p l_w^2 \frac{dU}{dY} \frac{dT}{dY} \]  
Thus, in terms of Prandtl's theory, the mixing length for heat was longer by a factor of \( \sqrt{2} \) than the mixing length for momentum. Since it is a frequent assumption, in work with free turbulence,\(^3\) that the mixing length is proportional to the width of the mixing region, this leads to a prediction by Taylor's theory that the observed width of the temperature mixing region will be wider than that observed for momentum.

Later, in 1942, Prandtl proposed a new theory of free turbulence to account for some discrepancies discovered in his original idea. Basing his theory on an assumption of similarity of velocity profiles with \( x \), he reasoned that the ratio of mixing length to mixing region
width is constant at similar points in the velocity profile,
\[ \frac{L}{b} = \text{const} \times \frac{y}{b} \]  (4-17)
Based on this, his theory could be expressed as
\[ \varepsilon_m = \chi b (u_{max} - u_{min}) \]  (4-18)
where \( \chi \) is an empirical constant and \( b \) is the width of the turbulent mixing region. A constant coefficient of turbulent conductivity can be expressed in a similar manner.
Prandtl's new theory, however, contained no assumptions relating the transfer of momentum to that of heat. Thus, on the basis of his theory, knowledge of the velocity profile is not sufficient to predict a corresponding temperature distribution.

The similarity theory proposed by Von Karman is another development worthy of note. His theory is based on ideas similar to Prandtl's original theory. Rather than a constant mixing length, he postulated a relationship for \( l \),
\[ l = \alpha \frac{(d\bar{u}/d\gamma)^2}{(d^2\bar{u}/d\gamma^2)} \]  (4-19)
where \( \alpha \) is a constant. This resulted in a relationship for the apparent shear stress,
\[ \tau_{app} = \rho \alpha^2 \frac{(d\bar{u}/d\gamma)^4}{(d^2\bar{u}/d\gamma^2)^2} \]  (4-20)
This relationship when applied to heat transfer, however, contributes no new information to the relationship of heat and momentum transfer.

In summary, the theories presented represent the
foremost in phenomenological developments. It may be noted, however, that two theories, Prandtl's mixing length and Taylor's vorticity transport theory, contain the only proposed relationships between heat and momentum transfer. Since the analysis to be presented does not give an exact solution, but attempts to relate the apparent shear stress to the apparent heat transfer, these two theories will be the only ones considered further.

B. Experimental Work

Since no experimental data is available considering cases where the velocity profile is dissimilar to that of the temperature profile, the data presented in this section will not be directly applicable to the present study. It is to be expected, nevertheless, that some of the data may be qualitatively applicable to the present case.

Several researchers who studied heat transfer in merging streams at different velocities and temperatures are cited by Abramovich. None of these researchers, however, considered the initial boundary layer. From the data presented, Abramovich draws several interesting conclusions. One of these conclusions is that the dimensionless velocity profiles are universal, that is, they don't vary with changes in the downstream direction. He also concludes that the dimensionless temperature profiles have
the same universality, and notes particularly that these profiles are unaffected by changes in the average velocities between the streams. Another observation is that the dimensionless temperature and velocity profiles do not coincide, and in fact the thermal mixing region is wider than that of momentum.

Ruden, another researcher cited by Abramovich, measured the mean velocity and temperature profiles in a heated axial-symmetric jet discharging into still air. He found the temperature profiles to be wider than those of velocity and to conform fairly closely to profiles predicted on the basis of Taylor's theory.

In a similar case summarized by Abramovich, Antonova measured the turbulence parameters, \( \sqrt{\bar{u}'^2} \), \( \sqrt{\bar{v}'^2} \), \( \sqrt{\bar{T}'^2} \), \( \langle u'v' \rangle \), \( \langle u'T \rangle \), and \( \langle v'T \rangle \) in an axial-symmetric jet. He was able to produce distributions of the longitudinal and lateral velocity fluctuations that could be reasonably described by dimensionless universal curves. It could be noted from plots of \( \sqrt{\bar{u}'^2} \) and \( \sqrt{\bar{v}'^2} \) that the longitudinal fluctuation was slightly greater than the lateral fluctuation. The other fluctuation terms shown showed considerable scatter. It could be noted, however, that the correlation coefficient for the two velocity components, \( R_{uv} = \frac{\bar{u'v'}}{\sqrt{\bar{u}'^2}\sqrt{\bar{v}'^2}} \) was considerably less than the correlation coefficient for the longitudinal component of velocity and temperature, \( R_{ut} = \frac{\bar{u'T}}{\sqrt{\bar{u}'^2}\sqrt{\bar{T}'^2}} \).
The experimental work in another flow case, that of the wake of a heated cylinder, is reviewed by Abramovich. This case may be of interest in the present situation because, although the temperature distribution is quite different from the present case, the velocity profile is quite similar. Abramovich, in comparing the velocity and temperature profiles determined by two different research groups, Reichardt and Fage and Falkner, found the data to give substantially better agreement with Taylor's theory than with Prandtl's theory. That is, the temperature profile showed a notably greater spread than the velocity profile.

Corrsin and Uberoi performed a thorough experimental study on a round, heated jet discharging into still air. A number of cases were considered, with the temperature difference between the jet and the atmosphere ranging from 15°C to 300°C. They again confirmed a more rapid spread of the thermal mixing region than that of momentum. They also found the velocity and temperature fluctuations to vary with their respective gradients in a manner which gave qualitative support to the phenomenological idea of a constant mixing length.

From the viewpoint of the present study, the most interesting point of Corrsin's study was a direct experimental calculation of a turbulent Prandtl number, defined
The study found the turbulent Prandtl number to be roughly 0.7 over most of the jet width. Unfortunately, near the jet axis and the outer edge of the jet the calculation became more or less indeterminate. Corrsin particularly notes that the turbulent Prandtl number is equal to the laminar Prandtl number of air within the probable accuracy of the experimental measurements. Other researchers, however, have noted turbulent Prandtl numbers ranging from 0.6 to 0.8 in different fluids with laminar Prandtl numbers ranging from 0.7 to 785.

Corrsin also considered the effect of density changes due to temperature on the spread of the jet. He found the hotter jets, that is the jets of lower density, to exhibit a more rapid spread of momentum. This effect of density has been confirmed by researchers working with gases of different densities, who have found that jets of fluid of lower density than the surrounding medium spread more rapidly than do jets of higher density. In addition, while working with two different gases, these workers have found the distribution of mass to spread more rapidly than momentum, in a manner similar to the spread of temperature. Thus, the more rapid spread of temperature than momentum appears to be due to the scalar
properties of temperature as opposed to the vector nature of momentum.

In conjunction with Kistler,\textsuperscript{15} Corrsin has also produced a statistical and experimental study on the nature of boundaries of turbulent flows. Though the statistical analysis is beyond the scope of this paper, the experimental results are of great interest in illuminating the nature of free turbulent flows. Evidence has been produced, both photographically and with the aid of a hot-wire anemometer, that the boundaries of turbulent mixing regions are far from uniform. In fact the picture is that of small turbulent eddies being convected by large eddies along the edge of the mixing region. This results in the boundary areas being turbulent for fractional periods of time, while in the other periods of time the areas are in the undisturbed flow. By defining an intermittency factor ($\mathcal{Y}$) which represents the fractional period of time for which the flow is turbulent at any locality, measurements can be made which show the nature of the turbulence. Using $\mathcal{Y}$, which has a value of zero for nonturbulent flow and a value of unity for flow which is continuously turbulent, Corrsin has shown that $\mathcal{Y}$ does not reach zero until it is substantially further from the center of flow than the boundary determined by the mean velocity. In addition, $\mathcal{Y}$ does not reach the value unity until a point very near
the centerline of the mixing region is reached. Thus, large areas of the mixing region may not be turbulent at any particular instant.

Corrsin also made studies of the intermittency in turbulent boundary layers. In these studies he found that the intermittency factor became unity near the boundary and approached zero near the free stream. Unfortunately, no data was taken for the case of a mixing region with an initial boundary layer. The effect of a boundary layer on the intermittency of a mixing region is therefore still open for discussion. The effects of intermittency on heat transfer were not considered.

In summary, Corrsin's study gives much insight to the nature of turbulence, but at present is not sufficiently developed to permit engineering application to turbulent flows. The phenomenological theories are still of value in the solution of problems, and further data may extend their value. For these reasons, the present study will attempt to develop the idea of a turbulent Prandtl number in the hope that it may be found to have practical importance.
Recall equations (4-11) and (4-13) which have been reduced to a form applicable to the present problems.

\[
\rho \left( \frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} \right) = -\rho \frac{\partial}{\partial y} (\bar{U}' \bar{V}') \tag{4-11}
\]

\[
\rho C_P \left( \frac{\partial \bar{T}}{\partial x} + \frac{\partial \bar{T}}{\partial y} \right) = -\rho C_P \frac{\partial}{\partial y} (\bar{V}' T') \tag{4-13}
\]

These equations have been reduced to the present form through the use of Prandtl's mixing length theory and an order of magnitude analysis.

Before application of these equations to the present case, a further justification should be presented.

First, the assumption that \( O(\bar{U}') \approx O(\bar{V}') \) was based on Prandtl's analysis. This appears to have been verified for numerous cases. Nevertheless it is felt that this assumption requires experimental verification in the present case.

Next, consider the assumption that \( \frac{\partial}{\partial x} (\bar{U}')^2 \ll \frac{\partial}{\partial y} (\bar{U}' \bar{V}') \). Due, at least partially, to the intermittent nature of the boundary of the mixing region, the terms \( (\bar{U}' \bar{V}') \) and \( (\bar{V}')^2 \) may be expected to pass through a significant range of intensity across the width of the mixing region. If the turbulence becomes more intermittent toward the edge of the mixing region, the measured magnitude of these terms may be expected to drop significantly, even if the intensity of the individual small eddies in this region is relatively
Thus the transverse gradient of \((\bar{U}'\bar{V}')\) is expected to be quite large. On the other hand, unless the flow rapidly becomes more intermittent as it proceeds downstream, or the small eddies rapidly die out, the longitudinal gradient of \((\bar{U}')^2\) should remain small. It may thus be expected that \(\frac{\partial}{\partial x}(\bar{U}')^2\) will be small in comparison with \(\frac{\partial}{\partial y}(\bar{U}'\bar{V}')\). One point where \(\frac{\partial}{\partial x}(\bar{U}')^2\) and \(\frac{\partial}{\partial y}(\bar{U}'\bar{V}')\) may be of the same order of magnitude would be close to the downstream end of the splitter plate, where the original solid boundary has become a free shear layer. Since a relatively thick boundary layer is anticipated, due to the nature of the wind tunnel, the effect of the initial sharp discontinuity should be at least partially nullified, and the assumption that \(\frac{\partial}{\partial x}(\bar{U}')^2 << \frac{\partial}{\partial y}(\bar{U}'\bar{V}')\) should be reliable in that region. Because of the rather loose nature of the preceding argument, experimental verification of the assumption is necessary.

Since the turbulence terms of the energy equation, \((\bar{U}'\bar{T}')\) and \((\bar{V}'\bar{T}')\), are expected to arise in the same manner as those of the equation of motion, a discussion of these terms will give the same result as the previous argument. That is, it may be anticipated that the transverse gradient of these terms will be significantly greater than the longitudinal gradient. Looking further at the temperature fluctuations, it can be seen that they arise when a fluid
at some temperature is convected by turbulent eddies into a region of different temperature. The temperature fluctuations should, therefore, be greatest where the temperature gradient and velocity fluctuations are greatest. The magnitude of the temperature fluctuations should be directly related to the magnitude of the velocity fluctuations. A comparison of the term \((\overline{T'})^2\) with the terms \((\overline{U'})^2\) and \((\overline{V'})^2\) is thus of some interest. The initial assumption that
\[
\frac{\partial}{\partial x}(\overline{U' T'}) \ll \frac{\partial}{\partial y}(\overline{V' T'})
\]
should also be confirmed by experiment.

Assuming the verity of equations (4-11) and (4-13), turn now to a discussion of the turbulent Prandtl number. By comparing the equations for turbulent motion with those for laminar flow in a similar case, it is felt that considerable appreciation may be gained for the turbulent Prandtl number.

The equations\(^7\) for a mixing region in laminar flow are
\[\rho (U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y}) = \frac{\partial}{\partial y} (\rho \frac{\partial U}{\partial y}) = \frac{\partial}{\partial y} (\overline{U'}) \tag{5-1}\]
\[\rho c_p (U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y}) = \frac{\partial}{\partial y} (K \frac{\partial T}{\partial y}) = \frac{\partial}{\partial y} (- q') \tag{5-2}\]
where the subscript \(l\) denotes the laminar case. Dimensional analysis of these equations has yielded the dimensionless group\(^3\) called the Prandtl number, \(Pr = \frac{\nu c_p}{K}\), which expresses the ratio of the molecular diffusion properties of momentum and heat. The relative thickness of momentum and thermal boundary layers has been found to be
a function of Pr in a number of laminar cases.

Now comparing equations (5-1) and (5-2) with (4-11) and (4-13), it can be seen that the left hand side of the turbulent equations are similar to the laminar equations, assuming that the mean time average values for the turbulent case are analogous to the values in the laminar case. Because of the similarity of the left hand sides of the equations, it seems reasonable to propose an analogy relating the right hand sides. Since the assumption has already been made, in Chapter III, that $\rho$ sees relatively small variation for the present case, the right hand sides of (4-11) and (4-13) may be written $\frac{\partial}{\partial y}(-\rho u'v')$ and $\frac{\partial}{\partial y}(-\rho c_p T'')$, respectively. At this point it should again be noted that in actuality, $\rho$ is a function of the temperature. If a mean value of $\rho$ is used, similar to the film values utilized in many boundary layer problems, and the temperature range is small, the error introduced should be small.

Continuing the analogy, note that the right hand sides of the laminar equations represent derivatives of the laminar shear stress and the laminar heat transfer. Because the mean values of velocity and temperature appear to show effects similar to those of greatly increased viscosity and conductivity, and because of the similarity of the left hand sides of the equations, it appears
logical to assume that the right hand sides of (4-10) and (4-13) are also derivatives of a shear stress and a heat transfer term, respectively. Utilizing these assumptions, the right hand sides of the turbulent equations may be written

\[-\rho \frac{\partial}{\partial y} (\bar{u}' \bar{v}') \approx \frac{\partial}{\partial y} (\rho \bar{u}' \bar{v}') \equiv \frac{\partial}{\partial y} (\tau_e) \]  
(5-3)

\[-\rho C_p \frac{\partial}{\partial y} (\bar{v}' \bar{T}') \approx \frac{\partial}{\partial y} (-\rho C_p \bar{v}' \bar{T}') \equiv \frac{\partial}{\partial y} (-\bar{q}_e) \]  
(5-4)

Thus the equations for the shear stress and the heat flow become

\[\tau_e \equiv -\rho (\bar{u}' \bar{v}') \]  
(5-5)

\[\bar{q}_e \equiv \rho C_p (\bar{v}' \bar{T}') \]  
(5-6)

for turbulent flow. In view of the assumptions made to obtain (5-5) and (5-6), it must be noted that the equations are more truly definitions than derived values. Nevertheless, they will have value in illuminating the physical relationship between heat and momentum transfer.

To extend the analogy it is useful to employ Boussinesq's concept of coefficients of eddy viscosity and eddy conductivity. Recall that these have been written

\[\tau_e = -\rho (\bar{u}' \bar{v}') = \epsilon_m \frac{\partial \bar{u}}{\partial y} \]  
(4-1)

and

\[\bar{q}_e = \rho C_p (\bar{v}' \bar{T}') = -\epsilon_q \frac{\partial \bar{T}}{\partial y} \]  
(4-2)

The terms \(\epsilon_m\) and \(\epsilon_q\) have now become analogous to the coefficients of viscosity and conductivity of the
laminar flow equations. Because they are functions of the flow field, rather than properties of the fluid, it is not reasonable to expect them to be constants. On the other hand, if heat and momentum are transferred by similar mechanisms in turbulent flow, their ratio may in fact be a constant. In analogy with the laminar Prandtl number, a definition of a turbulent Prandtl number may be given as

$$Pr_t \equiv \frac{C_p \rho \varepsilon_m}{\varepsilon_f} \quad (5-7)$$

Looking back at Prandtl's mixing length theory and at Taylor's vorticity transport theory it may now be observed that Prandtl's theory predicts a constant value, $Pr_t = 1$, and Taylor's hypothesis predicts $Pr_t = \sqrt{2}$. It can now be seen that the turbulent Prandtl number could be a constant, even though the mechanisms of heat and momentum transfer are not identical.

In an attempt to establish the physical significance of $Pr_t$, return to equations (4-11) and (4-13). By rewriting (4-10) using the Boussinesq concept of apparent shear, multiplying by $U$ and integrating with respect to $y$ over half the mixing region width, there results

$$\int_0^b \rho (U^2 \frac{\partial U}{\partial x} + U V \frac{\partial U}{\partial y}) dy = \int_0^b \varepsilon_m U \frac{\partial U}{\partial y} dy \quad (5-8)$$

Because of the expected symmetry of the velocity profile, it can be expected that an integration over the other half of the region will lead to the same results. It
may also be expected that the mean velocity at \( y = 0 \) will be a minimum, \( U_{\text{min}} \). At this point, for simplicity, consider the density to be constant, a supposition which as previously stated should not lead to large error. Now from the continuity equation a value for \( \overline{V} \) may be found.

\[
\overline{V} = -\int_0^y \frac{\partial \overline{U}}{\partial x} \, dy
\]  

(5-9)

Substituting (5-9), integrating by parts and making the crude assumption that \( \epsilon_m \) remains constant, the following equation may be obtained.

\[
\frac{1}{2} \rho \int_0^b \frac{\partial}{\partial x} (\overline{U}^3 - \overline{U}_b^3 \overline{U}) \, dy = \frac{1}{2} \rho \epsilon_m (u_b^2 - u_{\text{min}}^2)
\]

(5-10)

By application of Liebnitz's rule, \( 16 \) (5-8) may be restated as

\[
\frac{d}{dx} \int_0^b (\overline{U}^3 - \overline{U}_b^3 \overline{U}) \, dy = \epsilon_m (u_b^2 - u_{\text{min}}^2)
\]

or

\[
\epsilon_m = \frac{d}{dx} \int_0^b (\overline{U}^3 - \overline{U}_b^3 \overline{U}) \, dy = \frac{d/dx (I_m)}{u_b^2 - u_{\text{min}}^2}
\]

(5-11)

To obtain a similar expression for \( \epsilon_g \), turn to (4-13). By symmetry arguments it is apparent that the temperature at the centerline will be the average of the temperatures of each channel, \( T_{\text{ave}} \). By applying the concept of apparent heat transfer, multiplying by \( T \) and integrating over half the width of the thermal mixing region, there may be found

\[
\rho \int_0^{b_T} (\overline{U} \overline{T}_b^2 - \overline{U} \overline{T}_b^2) \, dy = \rho \epsilon_g \int_0^{b_T} \overline{T} \frac{\partial \overline{T}}{\partial y} \, dy
\]

(5-12)

where the assumptions of constant density and constant \( \epsilon \) have already been applied. Now, by again using (5-9)
and integrating by parts, (5-12) may be written as
\[
\frac{\rho}{2} \int_{0}^{br} \frac{\partial}{\partial x} (\bar{u} \bar{T}^2 - \bar{u} \bar{T}_b^2) \, dy = \frac{1}{2} \rho \varepsilon_q (T_b^2 - \bar{T}_{ave}^2) \tag{5-13}
\]
Application of Leibnitz's rule results in a restatement of (5-13).
\[
\frac{d}{dx} \left[ \int_{0}^{br} (\bar{u} \bar{T}^2 - \bar{u} \bar{T}_b^2) \, dy \right] = \varepsilon_q (T_b^2 - \bar{T}_{ave}^2) \tag{5-14}
\]
Equation (5-14) may be rearranged to yield an expression for \( \varepsilon_q \).
\[
\varepsilon_q = \frac{\frac{d}{dx} \int_{0}^{br} (\bar{u} \bar{T}^2 - \bar{u} \bar{T}_b^2) \, dy}{T_b^2 - \bar{T}_{ave}^2} = \frac{d}{dx} \left( \frac{E_{m}}{T_b^2 - \bar{T}_{ave}^2} \right) \tag{5-15}
\]
Consider now the significance of (5-11) and (5-15). If a proper function for the velocity profile can be found, the integral of (5-11) may be evaluated to yield \( b \) as a function of \( x \). Similarly, substitution of appropriate velocity and temperature functions will give a thermal mixing region width from (5-15). The assumptions that \( \varepsilon_m \) and \( \varepsilon_q \) are constant are so restrictive, however, that these results may be expected to give only qualitatively correct results.

Returning to the turbulent Prandtl number, substitute (5-11) and (5-15) into the definition of \( Pr_t \), equation (5-7). An expression relating the relative widths of the thermal and momentum mixing regions results.
\[
Pr_t = \frac{C_P (T_b^2 - \bar{T}_{ave}^2)}{U_b - U_{\bar{u}_{m,m}}} \frac{d/dx (I_m)}{d/dx (I_q)} \tag{5-16}
\]
Though this expression deals with just one half of the mixing region, a similar treatment of the other half of the region will give the same result. It should
be noted that the equations leading to this expression for $Pr_c$ may be integrated over the full width of the mixing region if the velocities in each channel are not equal. It should also be noted that while (5-8) is an expression of the mechanical energy of the flow, there is no particular physical significance to (5-12). Multiplying by $\overline{v}$ to obtain (5-12) may be simply regarded as a device to retain $\epsilon_\gamma$ in the equation. Though this analysis has been qualified by several severe assumptions, it is hoped that the analysis is indicative of the possible value of the turbulent Prandtl number.

If $Pr_c$ may now be confirmed to remain constant, a valuable physical relationship can be established. In order to experimentally determine $Pr_c$, turn again to the definition of $Pr_c$ and substitute the values of $\epsilon_m$ and $\epsilon_\gamma$ from (4-1) and (4-2). The following equation results.

$$Pr_c = \frac{(\overline{w'u'})}{(\overline{v'w'})} \frac{\partial \overline{v}}{\partial y}$$

(5-8)

Using this equation $Pr_c$ may be experimentally determined from point to point in a flow field.

As summarized in Chapter IV, Corrsin and Uberoi have performed an experimental verification of $Pr_c$ for a heated, axi-symmetric jet discharging into still air. They found $Pr_c$ to approximate a constant at 0.7. If this value can be extended to other cases, a valuable relationship may be extended to the analysis of turbulent flows.
CHAPTER VI

THREEORY OF MEASUREMENTS

Further reductions of equations (4-12) and (4-13) describing the present flow problem require experimental information about the turbulent fluctuation terms \((\overline{u'}^2), (\overline{v'}^2), (\overline{T'}^2), (\overline{u'v'}), (\overline{u'T'}), \) and \((\overline{v'T'})\). This chapter will show the development of equations by which these terms may be measured with the aid of hot-wire anemometry equipment.

Corrsin\(^1\) has shown that the voltage response of a hot wire to velocity and temperature fluctuations may be expressed in the form

\[
E' = A_1 u' + B_1 T' \quad (6-1)
\]

where \(B\) and \(A\) are sensitivities to temperature and velocity fluctuations, respectively. The present problem consists of adapting (6-1) to the compensated constant current anemometer which was available for experimental use.

The approach taken was to (A) determine the form of the electrical output of the anemometer in terms of the heat transfer coefficient of the wire and (B) isolate the desired fluctuating flow properties in the heat transfer coefficient.

(A) - Electrical Output of the Anemometer

The sensor of a hot wire anemometer is essentially a long thin wire which is simultaneously heated by a constant current (hence no fluctuations of current will occur)
and cooled by the moving air in which it is immersed. The obvious approach to the problem is application of an energy balance equation\textsuperscript{17} to the wire.

\[
(\text{energy in}) = (\text{energy out}) + (\text{energy stored})
\]

\[I^2 R = (T_w - T) H + \psi \frac{dT}{dt} \tag{6-2}\]

For the tungsten wire used, it has been experimentally verified that the wire resistance, over the range of temperatures encountered, varies linearly with the wire temperature.\textsuperscript{18} Thus the equation for the relationship between resistance and temperature may be expressed

\[
\frac{T}{T_*} = \frac{R_o + d}{R_o^* + d} \tag{6-3}
\]

where the ( )\textsuperscript{*} represents an arbitrary reference point.

Utilizing (6-3), (6-1) may be rewritten to give

\[
I^2 R = \frac{T^*}{R_o^* + d} (R - R_o) H + \frac{T^*}{R_o^* + d} \psi \frac{dR}{dt} \tag{6-4a}
\]

or by defining a constant \( N = \frac{R_o^* + d}{T^*} \)

\[
N I^2 R = (R - R_o) H + \psi \frac{dR}{dt} \tag{6-4b}
\]

Assuming now that the flow properties are fluctuating, a fluctuation in wire resistance and heat transfer coefficient will be expected. These may be written in a manner similar to (3-10), \( R = \bar{R} + \bar{R}' \), \( R_o = \bar{R}_o + \bar{R}_o' \), and \( H = \bar{H} + \bar{H}' \). By assuming the relative magnitude of these fluctuations to be small, they may be substituted into (6-4b) and the resulting higher order terms can be neglected, thus giving

\[
N I^2 (\bar{R} + \bar{R}') = (\bar{R} - \bar{R}_o) \bar{H} + (\bar{R}' - \bar{R}_o') \bar{H} + (\bar{R} - \bar{R}_o) \bar{H}' + \psi \frac{dR'}{dt} \tag{6-5a}
\]

Taking the mean time average (m.t.a.), (6-5a) becomes
\[ N I^2 R = (\bar{R} - R_0) \bar{H} \]  \hspace{1cm} (6-5b)

Subtracting (6-5b) from (6-5a) and noting that \( IR' = E' \), (6-5a) may be written after some rearrangement

\[ E' + \frac{\psi}{H - NI^2} \frac{d}{dt} (E') = \frac{I}{H - NI^2} \left[ R_0 \bar{H} - (\bar{R} - R_0) \bar{H}' \right] \]  \hspace{1cm} (6-6)

A compensated a.c. amplifier may be used to give an output proportional to the left hand side of (6-6).

The output of this amplifier will be in the form

\[ E_s = K_i (E' + \lambda \frac{dE'}{dt}) \]  \hspace{1cm} (6-7)

In order to solve for the amplifier constant, \( K_i \), and to adjust the time constant \( \lambda \) to match the wire time constant, a method using a square wave current has been devised.

Applying a square wave current to the hot wire, \( I_s = I + i \).

This fluctuating current will induce a fluctuating resistance, \( R = \bar{R} + R'_s \), due to heating effects. If at the same time the wire is immersed in still air at constant temperature \( R_0 = \bar{R}_0 \) and \( H = \bar{H} \). Substituting these terms in (6-4b), taking the m.t.a., subtracting and noting that \( I R'_s = E'_s \), an equation similar to (6-6) can be formed.

\[ E'_s + \frac{\psi}{H - NI^2} \frac{dE'_s}{dt} = \frac{2NI^2 i R}{H - NI^2} \]  \hspace{1cm} (6-8)

Now if the amplifier compensation is adjusted so that a square wave is the output when a square wave is the input, the time constant of the amplifier and the time constant of the wire will be equal, \( \lambda = \frac{\psi}{H - NI^2} \).

Thus, by writing

\[ E_s = K_i (E'_s + \lambda \frac{dE'_s}{dt}) = K_i (E'_s + \frac{\psi}{H - NI^2} \frac{dE'_s}{dt}) = K_i \frac{2NI^2 i R}{H - NI^2} \]
the amplifier constant, $K_i$, may be found.

$$K_i = \varepsilon_s \frac{H - NI^2}{2 NI^2 \zeta R}$$

(6-9)

Returning to (6-7) and assuming that the amplifier is properly compensated, (6-9) may be used to substitute for $K_i$.

$$C_f = \varepsilon_s \frac{H - NI^2}{2 NI^2 \zeta R} \left[ E' + \frac{\psi}{H - NI^2} \frac{d}{dt}(E') \right]$$

(6-10)

Utilizing (6-5b) and (6-6), defining $\sigma = \frac{R}{R_0}$, and defining $e = \frac{\varepsilon}{I} \frac{C_f}{C_s}$, (6-10) may be written in a convenient form.

$$C = \frac{I}{I} \frac{C_f}{C_s} = \frac{1}{2(\sigma - 1)} \frac{R_0}{R_0} - \frac{I}{2} \frac{H'}{H}$$

(6-11)

The term on the left of (6-11) is made up of electrical terms which may be measured quite readily.

With the aid of a true root mean square voltmeter the terms on the left of (6-11), which represent the output of the hot-wire anemometer, may now be measured. Since two wires will be used, it will be convenient to let the subscripts $()_1$ and $( )_2$ represent the terms arising from the individual wires, and the subscripts $( )_{1+2}$ and $( )_{1-2}$ represent those arising from situations where the output of the two wires have been added or subtracted by the sum difference unit.

With the rms voltmeter, the terms $M_n = \sqrt{\varepsilon_n^2} K_v$, $M_{n+v} = K_v \sqrt{(\varepsilon_f + \varepsilon_n)^2}$, and $M_{n+v+s} = K_v \sqrt{(\varepsilon_f + \varepsilon_n + \varepsilon_s)^2}$ can be measured. Here $M_n$ represents the rms value of the anemometer output with the wire unheated, $M_{n+v}$ is the value with the heated wire immersed in the stream, $M_{n+v+s}$ is the
value with the square wave superimposed, and \( K_v \) is a voltmeter constant. By assuming that \( e_{s_1} = e_{s_2} \), a condition which involves the use of wires with well matched properties, the following equations may be written

\[
e_{i}^{2} = \left( \frac{i}{I} \right)^{2} \frac{\left( M_{n+v} \right)^{2} - \left( M_{n} \right)^{2}}{\left( M_{n+v+s} \right)^{2} - \left( M_{n+v} \right)^{2}} \tag{6-12}
\]

\[
e_{2}^{2} = \left( \frac{i}{I} \right)^{2} \frac{\left( M_{n+v} \right)^{2} - \left( M_{n} \right)^{2}}{\left( M_{n+v+s} \right)^{2} - \left( M_{n+v} \right)^{2}} \tag{6-13}
\]

\[
(e_{1}+e_{2})^{2} = \left( \frac{i}{I} \right)^{2} \frac{\left( M_{n+v} \right)^{2} + \left( M_{n} \right)^{2} - \left( M_{n} \right)^{2}}{\left( M_{n+v+s} \right)^{2} - \left( M_{n+v} \right)^{2}} = \left( \frac{i}{I} \right)^{2} \frac{\left( M_{n+v} \right)^{2} + \left( M_{n} \right)^{2} - \left( M_{n} \right)^{2}}{\left( M_{n+v+s} \right)^{2} - \left( M_{n+v} \right)^{2}} \tag{6-14}
\]

\[
(e_{1}-e_{2})^{2} = \left( \frac{i}{I} \right)^{2} \frac{\left( M_{n+v} \right)^{2} - \left( M_{n} \right)^{2} - \left( M_{n} \right)^{2}}{\left( M_{n+v+s} \right)^{2} - \left( M_{n+v} \right)^{2}} = \left( \frac{i}{I} \right)^{2} \frac{\left( M_{n+v} \right)^{2} - \left( M_{n} \right)^{2} - \left( M_{n} \right)^{2}}{\left( M_{n+v+s} \right)^{2} - \left( M_{n+v} \right)^{2}} \tag{6-15}
\]

With the electrical terms on the left of (6-11) now in a form which may readily be measured, the remaining task is to describe \( \frac{R_{o}'}{R_{o}} \) and \( \frac{H'}{H} \) in terms of the desired fluctuating values of \( u, v, \) and \( T \).

(B) - Description of Fluctuating Flow Properties

The description of the fluctuating terms on the right of (6-11) requires the use of a semi-empirical equation describing the heat transfer coefficient, \( H \). Common procedure in anemometry work has been to use King's formula\(^{17}\)

\[
Nu = A + B Re^{\nu_{2}}
\]

This formula has been verified for steady flow, but no way has been found to confirm it for the rapidly fluctuating velocities and temperatures occurring in turbulence.
Nevertheless, it appears to be the most satisfactory equation available for anemometry work.

Steady-state empiricism has led to the following form of King's formula,

\[ H = \left( \frac{nD}{2} \right)^2 \frac{K}{\alpha \lambda} \left[ 0.3 + 0.51 \left( \frac{\mu D}{\rho} \right)^{1/2} \right]. \] (6-16)

Now expressing the air properties in terms of temperature, \( K = K^* \left( \frac{T}{T^*} \right)^{86} \) and \( \mu = \mu^* \left( \frac{T}{T^*} \right)^{71} \), and noting that for small temperature ranges \( \rho \approx \rho^* \frac{T^*}{T} \), (6-16) may be rewritten

\[ H = 0.3 \left( \frac{nD}{2} \right)^2 \left( \frac{K^*}{\alpha \lambda} \right) \left( \frac{T}{T^*} \right)^{86} \left[ 1 + \frac{0.51}{3} \left( \frac{\rho^* D}{\mu^*} \right)^{1/2} \left( \frac{V}{T} \right)^{-71} \right]^{1/2} \] (6-17)

Noting now that \( V \) is the component of velocity normal to the hot wire an expression can be formulated to give the normal velocity as a function of the free stream velocity \( (U) \), the angle \( (\beta) \) between the free stream direction and the wire, and an empirical constant \( (m) \) which corrects for the effects of a finite wire. Now, defining

\[ C = U \left( \sin \beta \right)^{1/m} \]

\[ C_0 = \sqrt{\frac{3}{51}} \frac{\mu^*}{\rho^* D} \]

\[ S = 0.3 \left( \frac{nD}{2} \right)^2 \left( \frac{K^*}{\alpha \lambda} \right) \]

a revision of (6-13) can be made.

\[ H = S \left( \frac{T}{T^*} \right)^{86} \left[ 1 + \left( \frac{C}{C_0} \right)^{1/2} \right] \] (6-18)

For further convenience, several more definitions
will be made.

\[ f = 5 \left( \frac{T}{T_a} \right)^{.86} \]
\[ g = 1 + \left( \frac{C}{C_o} \right)^{.5} \]
\[ H = f \cdot g \]

Now, assuming that the fluctuations in H are small, which implies a small fluctuation in both temperature and velocity, H' may be approximated by dH, thus giving

\[ \frac{H'}{H} = \frac{dg}{g} + \frac{df}{f} \tag{6-19} \]

df is now given by

\[ df = .86 \cdot 5 \left( \frac{T}{T_a} \right)^{.86} \frac{dT}{T} \]

and since the fluctuations in T are assumed small dT can be written as T'. Dividing by the expression for T gives

\[ \frac{df}{f} = .86 \cdot \frac{T'}{T} \tag{6-20} \]

applying a similar treatment to g

\[ \frac{dg}{g} = \left( \frac{C}{C_o} \right)^{.5} \frac{1}{2 \left[ 1 + \left( \frac{C}{C_o} \right)^{.5} \right]} \frac{dC}{C} = \frac{1 - (I_o/I)^2}{2} \frac{dC}{C} \]

since \( \left( \frac{C}{C_o} \right)^{.5} \frac{1}{2 \left[ 1 + \left( \frac{C}{C_o} \right)^{.5} \right]} = \frac{1}{2} \left[ 1 - (I_o/I)^2 \right] \)

C may also be handled in the same manner to give

\[ \frac{dC}{C} = \frac{U'}{U} + \frac{V'}{V} \cdot \frac{1}{m \tan \beta} - 1.71 \frac{T'}{T} \tag{6-21} \]

Substituting (6-20) and (6-21) into (6-19) gives

\[ \frac{H'}{H} = 0.86 \cdot \frac{T'}{T} + \frac{1 - (I_o/I)^2}{2} \left[ \frac{U'}{U} + \frac{V'}{V} \frac{1}{m \tan \beta} - 1.71 \frac{T'}{T} \right] \tag{6-22} \]

completing the analysis of the fluctuating heat transfer coefficient.

Turning now to the fluctuating resistance term
of (6-11) it can be noted that
\[
\frac{T'}{T} = \frac{R_o + d}{R_o' + d}
\]
and
\[
\frac{T + T'}{T'} = \frac{R_o + R_o' + d}{R_o' + d}
\]
By first subtracting the m.t.a. equation and then dividing by it, the expression
\[
\frac{R_o'}{R_o} = \frac{R_o + d}{R_o}. \frac{T'}{T}
\]
may be found.

This portion of the analysis may now be completed by substituting (6-22) and (6-23) into (6-11), resulting in the equation
\[
E = \left\{ \frac{1.86 - 0.86\sigma + \frac{d}{4}}{2(\sigma - 1)} \right\} + 0.427 \left( 1 - \frac{(\frac{T'}{T})^2}{1} \right) \frac{T'}{T} - \frac{1}{4} \left[ \frac{u'}{u} + \frac{\nu'}{u} \right] m_{Tm} \beta
\]
By setting \( A \) equal to \(-\frac{1}{4} \left[ 1 - \left( \frac{T'}{T} \right)^2 \right]\) and setting \( B \) equal to the coefficient of \( \frac{T'}{T} \), (6-24) may be written
\[
E_1 = A_1 \left[ \frac{u'}{u} + \frac{1}{m_{Tm}\beta} \frac{\nu'}{u} \right] + B_1 \frac{T'}{T}
\]
or
\[
E_2 = A_2 \left[ \frac{u'}{u} - \frac{1}{m_{Tm}\beta} \frac{\nu'}{u} \right] + B_2 \frac{T'}{T}
\]
where the sign change depends on the orientation of the wire.\(^{17}\) That is, if two wires are placed in a stream with the flow bisecting the angle between them, the sign of the \( \frac{\nu'}{u} \) term will be positive for one wire and negative for the other.

If it is now assumed that the two wires are well matched, so that \( A_1 = A_2 = A \) and \( B_1 = B_2 = B \), the desired turbulent fluctuation terms may be found by squaring (6-25) and (6-26) and taking their mean time average.\(^{19}\) The
resulting equations are
\[
\bar{E}_1^2 = A^2 \frac{(u')^2}{u^2} + \frac{A^2}{m^2 \cdot T \cdot \sigma^2} \frac{(v')^2}{v^2} + B^2 \frac{(T')^2}{T^2} \\
+ 2 \frac{A^2}{m \cdot T \cdot \sigma^2} \frac{(u'v')}{u^2} - 2AB \frac{(u'T')}{uT} - \frac{2AB}{m \cdot T \cdot \sigma^2} \frac{(v'T')}{uT} \tag{6-27}
\]
\[
\bar{E}_2^2 = A^2 \frac{(u')^2}{u^2} + \frac{A^2}{m^2 \cdot T \cdot \sigma^2} \frac{(v')^2}{v^2} + B^2 \frac{(T')^2}{T^2} \\
- 2 \frac{A^2}{m \cdot T \cdot \sigma^2} \frac{(u'v')}{u^2} - 2AB \frac{(u'T')}{uT} + \frac{2AB}{m \cdot T \cdot \sigma^2} \frac{(v'T')}{uT} \tag{6-28}
\]

By summing (6-25) and (6-26), squaring, and taking the mean time average of the resulting equation, the following equation results
\[
\overline{(E_1 - E_2)^2} = 4A^2 \frac{(u')^2}{u^2} - 4AB \frac{(u'T')}{uT} + 4B^2 \frac{(T')^2}{T^2} \tag{6-29}
\]
Treating the difference of (6-25) and (6-26) in a like manner, another equation is
\[
\overline{(E_1 - E_2)^2} = \frac{4A^2}{m^2 \cdot T \cdot \sigma^2} \frac{(v')^2}{v^2} \tag{6-30}
\]

Equations (6-27) through (6-30) represent four equations with six unknowns. By changing the resistance ratio, \(\sigma\), at which measurements are taken, the coefficients \(A\) and \(B\) will be altered. Thus by taking measurements at two different values of \(\sigma\), a system of eight equations with six unknowns will occur. This system may readily be solved. Since the solution is rather lengthy, a computer program (see Appendix B) was written to solve the equations for \(\frac{(u')^2}{u^2}, \frac{(v')^2}{v^2}, \frac{(T')^2}{T^2}, \frac{(u'v')}{u^2}, \frac{(u'T')}{uT}\), and \(\frac{(v'T')}{uT}\). The terms \(\bar{u}\) and \(\bar{T}\) may then be found from experimental curves of velocity versus wire current and temperature versus wire resistance.
A brief summary of assumptions made in this analysis should be made, with an eye toward their application to the present problem. The validity of King's equation in fluctuating flow was the most basic assumption. As previously stated, it has not been proven, but appears to give good results. The next assumption is that the fluctuations involved, both electrical and turbulent, will be small. This permitted the fluctuations to be represented as derivatives and the higher order terms to be neglected. Another critical restriction was the assumption of well matched wires, an assumption necessary to allow the addition or subtraction of the output of the two wires. These last assumptions may be experimentally verified for application to the present problem.
Prior to a discussion of the experimental results, a brief review of the procedure used in the gathering of data is in order. The importance of the procedure used lies in its possible effects on the data gathered.

One flow situation was studied, that of equal average flow velocities in both channels and a single average temperature differential between channels. The average channel flow velocity used was 36 feet per second and the temperature difference was 38°F. However, due to the lengthy nature of the measurements, it was impossible to maintain these velocities and temperature differentials constant over the complete period of data accumulation. As a result, the velocity varied from 33 to 36 feet per second in a channel while a differential in average velocities of up to 4 feet per second was observed between the channels. The temperature difference between the channels ranged from 36°F to 39°F.

By assuming a linear relationship between the resistance of the hot wire and the flow temperature, it was possible to measure the temperature directly with the anemometer. This relationship between resistance and temperature was experimentally determined, and a plot of resistance versus temperature approximated a straight line.
quite closely. Mean velocities were determined by using a linear relationship between the square of the wire current and the square root of the velocity, a relationship which is widely used in anemometry.

One peculiarity of the hot wire probe should be noted. The Flow Corporation booklet states that the wire cooling constant should approach unity. The cooling constant determined for the wires was found to be approximately 2.0. One possible reason for this was a slight bow in one of the wires. This defect was noted, but, because of the difficulty of mounting wires, was neglected. Possibly, the determination of \((\overline{v'})^2\) and \((\overline{u'v'})\) could be seriously affected as a result. The wire properties were closely matched, within 5%. The equations developed in Chapter VI should thus be applicable.

Because of the nature of its construction, the wind tunnel itself may contribute substantially to experimental error. The possibilities of three-dimensional flow and induced vibrations have already been discussed by Kascoutas and will not be pursued further. It must be noted that the high levels of turbulent intensity found by Kascoutas were greatly reduced in the present study by removal of screens which were intended to produce a thickened initial boundary layer. The present free stream intensities are about 1%, which is still quite high when
compared with the 0.01 - 0.1\% recommended by Kovasznay.\textsuperscript{20}

Keeping these factors in mind, refer now to the figures of Appendix A with the hope that indicative results can be observed.

Consider the order of magnitude analysis made in Chapter IV. The entire analysis was based on the assumption that $O(y) \ll O(x)$. Referring to Figure 3, it can be seen that over a distance of 15 inches in the x direction the width (y direction) of the momentum mixing region ranges from $1\frac{1}{2}$ inches to 3 inches, while the width of the energy mixing region ranged from $\frac{1}{2}$ to 2 inches. Since the rate of spread of the region is not rapid, the order of magnitude assumption appears valid.

The second assumption of the analysis was $O(\overline{u}^2) = O(\overline{v}^2)$. Reference to Figure 4 and Figure 5 indicates that $\overline{(v^2)}$ is greater than $\overline{(u^2)}$ by factors ranging from 2 to 10. The assumption, therefore, remains in doubt. There is evidence for most jets that $\overline{u^2}$ is slightly larger than $\overline{v^2}$. As mentioned previously, the high value of the wire cooling constant could affect the measurement of $\overline{(v^2)}$ adversely. Since the $\overline{(v^2)}$ term is proportional to the cooling constant, this appears likely. One other possibility remains. The hot-wire probe was physically attached to the wall of the wind tunnel. Thus transverse vibrations of the tunnel could cause a motion
of the probe resulting in an exaggerated value of $(\overline{v'})^2$.

A third assumption requiring verification was the supposition that $\frac{\partial}{\partial x} (\overline{u'})^2 < < \frac{\partial}{\partial y} (\overline{u'v'})$. A brief perusal of Figures 3 and 4 shows the curve of $\frac{(\overline{u'})^2}{U_m^2}$ to be quite regular and smooth, while the plot of $\frac{(\overline{u'v'})}{U_m^2}$ undergoes a rather sharp change at the centerline. Figure 4 shows $\frac{(\overline{u'})^2}{U_m^2}$ to vary by a maximum of $4 \times 10^{-4}$ over the total distance of 15 inches. Figure 6 indicates a minimum range of $\frac{(\overline{u'v'})}{U_m^2}$ to be $4 \times 10^{-4}$ across the width of the mixing region, on a curve which has two inflection points. Based on this, it is logical to conclude for the present case that $\frac{\partial}{\partial x} (\overline{u'})^2$ is in fact much less than $\frac{\partial}{\partial y} (\overline{u'v'})$ and may be dropped from the equation of motion.

The primary supposition utilized in the reduction of the energy equation was that $O(\overline{u'T'}) = O(\overline{v'T'})$, which resulted in $\frac{\partial}{\partial x} (\overline{u'T'}) < < \frac{\partial}{\partial y} (\overline{v'T'})$. Figures 7 and 8 show $\frac{(\overline{u'T'})}{(U \Delta T)_{max}}$ to be somewhat less than $\frac{(\overline{v'T'})}{(U \Delta T)_{max}}$, a result which would support the conclusion that $\frac{\partial}{\partial x} (\overline{u'T'}) < < \frac{\partial}{\partial y} (\overline{v'T'})$. Doubt may be cast on this result, however, by recalling that the measurement of $(\overline{v'})^2$ is somewhat questionable. Nevertheless, Figure 8 indicates a minimum change in $\frac{(\overline{v'T'})}{(U \Delta T)_{max}}$ of $2 \times 10^{-4}$ over the width of the mixing region, while Figure 7 shows a maximum change of $\frac{(\overline{u'T'})}{(U \Delta T)_{max}}$ of $3 \times 10^{-5}$ over the total
length studied. Thus a conclusion that \( \frac{\partial}{\partial x}(u'T') \) is less than \( \frac{\partial}{\partial y}(v'T') \) by an order of magnitude seems prudent, and the resulting equation may be deemed at least qualitatively correct.

Before proceeding further, it is felt that several peculiarities in the data should be noted. A glance at Figure 4 shows a marked skewness of the curve of \( \frac{\bar{u}^2}{u_{max}^2} \) toward the right, that is, toward the unheated channel. Another noteworthy observation is that the curves of transverse velocity fluctuation, temperature fluctuation, mean velocity, and mean temperature (Figures 5, 9, 10 and 11) do not show a similar skewness. If this is an indication of some physical phenomenon, the work of other researchers, studied by this author, would not have shown it because of the symmetry of the cases studied. Unfortunately, the author knows of no explanation for this peculiarity.

Figure 3, the mixing region boundaries, is worthy of some discussion. Because of the difficulty in establishing the boundaries, the curves should be regarded as approximations, at best. Since the initial momentum boundary layer was considerably thicker than that for temperature, the graph will not show a thermal mixing region wider than the momentum region. Nevertheless, it appears that the thermal boundary is spreading more rapidly than the momentum boundary. Since, as previously
noted, variance in the flow conditions was introduced from day to day, the mixing region was altered to the extent that no observation of the functional nature of the boundary would be valid. Since only one flow setting was studied, with a single temperature differential between channels, effects of density on mixing region thickness could not be observed.

Consider now Figures 6, 7 and 8 \( \frac{u'u'}{u_{max}^2} \), \( \frac{u'T'}{(u \Delta T)_{max}} \) and \( \frac{v'T'}{(u \Delta T)_{max}} \). An explanation of the different shapes of these curves is in order. First, the curve \( \frac{u'u'}{u_{max}^2} \) shows two inflection points, resulting in negative values over half the mixing region and positive values over the other half. According to momentum transfer theory, the sign of this term should be a function of the sign of the velocity gradient. Thus the change in signs at the centerline occurs. Because of the incomplete nature of the experimental data, the actual values of \( \frac{u'u'}{u_{max}^2} \) near the centerline remain in doubt. There is no way of telling where the maximum will occur without more complete measurements. Now note that the terms \( \frac{u'T'}{(u \Delta T)_{max}} \) and \( \frac{v'T'}{(u \Delta T)_{max}} \) are of the same sign. It should be expected that \( \frac{v'T'}{(u \Delta T)_{max}} \) is positive along the negative temperature gradient. Since the longitudinal temperature gradient changes from negative to positive as the transverse location moves from the
heated channel, and \((\overline{u'}\overline{v'})\) also changes from negative to positive the \((\overline{u'}\overline{T'})\) should be positive across the full width of the channel.

Before discussing the turbulent Prandtl number, note should be made of the difficulties involved in its calculation. Looking at equation (5-8),

\[
Pr_\tau = \frac{(\overline{u'}\overline{v'})}{(\overline{v'}\overline{T'})} \frac{\partial \overline{T}}{\partial y} \frac{\partial u}{\partial y}
\]  

(5-8)

It becomes obvious that an experimental calculation will become indeterminate at points where the velocity gradient, \(\frac{\partial u}{\partial y}\), approaches zero. In the present case this situation occurred at the outer edges of the mixing region and at the center. As a result, considerable variations in the calculated value of \(Pr_\tau\) may be expected near the outer edges and the center of the mixing region.

Now, looking at Figure 12, which is a plot of \(Pr_\tau\) versus \(y\), it is obvious that \(Pr_\tau\) does not have the constant value desired, even when the outer edges and the center of the mixing region are neglected. As has been discussed, some possibility of experimental error is present. Nevertheless, the other properties of the flow field appear to be at least qualitatively in agreement with the experimental work of other researchers. Consequently, the calculated values of \(Pr_\tau\) should also be qualitatively correct. Since no distinguishable pattern appears in Figure 12, it is felt that the concept of a
turbulent Prandtl number has no valid physical basis in the present flow situation.
CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

A. Conclusions

From the experimental data, the following results were obtained:

From mean velocity and temperature data, it could be concluded that the order of magnitude assumption, \( O(y)^{-1} \ll O(x) \), was valid. A more rapid spread of thermal mixing region boundaries than momentum boundaries seemed apparent. No conclusions concerning a functional nature of the boundaries could be drawn, however.

The measurement of fluctuating components of velocity and temperature produced a number of conclusions. Though the analytical assumption that \( O(\overline{u'^2}) = O(\overline{v'^2}) \) could not be definitely affirmed, the experimental evidence that \( \frac{\partial}{\partial x} (\overline{u'^2}) < < \frac{\partial}{\partial y} (\overline{u'v'}) \) and \( \frac{\partial}{\partial x} (\overline{u'T'}) < < \frac{\partial}{\partial y} (\overline{v'T'}) \) seemed conclusive. As a result, the equations of motion and energy, (4-11) and (4-13), proposed for the present flow situation appear valid.

The turbulent Prandtl number was not found to be constant or even approach a simply expressible functional behavior. Thus for the present problem, it appears that the turbulent Prandtl number does not give a useful expression relating the velocity and temperature profiles or their respective boundaries.
B. Recommendations

The wind tunnel appeared to give satisfactory velocity and temperature profiles for the purposes of the present studies. Because the measured \((v')^2\) term appeared high, however, it is felt that a modification of the probe mountings should be made to isolate the probe from the vibrations of the tunnel wall. This could be done by constructing a separate stand for the probe mountings and sealing the tunnel wall with a material such as foam rubber.

A number of problems similar to the present case could be studied in the future. If additional heat could be added to the air stream, the versatility of the tunnel could be extended, permitting the study of heat transfer in turbulent flows at velocities higher than those studied in the present situation. More advanced studies could be undertaken to consider the statistical nature of turbulent flows. From the present viewpoint, a study considering the intermittent nature of the turbulent mixing region would be of interest.
BIBLIOGRAPHY


(20) Kovasznay, L., "Should We Still Use Hot Wires?," Presented at the International Symposium on Hot Wire Anemometry, University of Maryland, College Park, Maryland, March, 1967.
Figure 3. Approximate Mixing Region Boundaries
Figure 4. Longitudinal Fluctuations
Figure 5. Transverse Fluctuations
Figure 6. Apparent Shear Stress
Figure 7. Longitudinal Apparent Heat Transfer
Figure 8. Lateral Apparent Heat Transfer
\[
\frac{(T')^2}{(\Delta T_{\text{max}})} \times 10^3
\]

Figure 9. Temperature Fluctuations
Figure 10. Mean Velocity
Key
- 1.0 = x
- 2.0 = △
- 3.0 = △
- 4.0 = □
- 6.0 = □
- 9.0 = ∇
- 12.0 = ∇
- 15.0 = ○

Figure 12

Turbulent Prandtl Number

Prandtl's Theory

Taylor's Theory
Because of the lengthy nature of a solution to equations (6-27), (6-28), (6-29) and (6-30), a solution was programmed enabling the use of an IBM 1620 computer. The following is the program solving for the values \( \frac{\nu'}{\nu} \), \( \frac{\tau'}{\tau} \), \( \frac{\tau'}{\tau} \), \( \frac{\nu'}{\nu} \), and \( \nu' \).
11 \( X4A = \frac{(V4A{-}\cdot1H{-}2 \cdot D1A{-}\cdot2 \cdot D2A{-}\cdot2}{(S1A{-}\cdot2 + S2A{-}\cdot2 - V1A{-}\cdot2 - V2A{-}\cdot2)} \)

12 \( E4A = 2 \cdot \frac{(2 \cdot FA)/(F1A + F2A)^{-2} \cdot X4A}{(S1B{-}\cdot2 + S2B{-}\cdot2 - V1B{-}\cdot2 - V2B{-}\cdot2)} \)

13 \( X4B = \frac{(V4B{-}\cdot1H{-}2 \cdot D1B{-}\cdot2 \cdot D2B{-}\cdot2)}{(S1B{-}\cdot2 + S2B{-}\cdot2 - V1B{-}\cdot2 - V2B{-}\cdot2)} \)

14 \( E4B = 2 \cdot \frac{(2 \cdot FB)/(F1B + F2B)^{-2} \cdot X4B}{(S1B{-}\cdot2 + S2B{-}\cdot2 - V1B{-}\cdot2 - V2B{-}\cdot2)} \)

15 \( AA = (1 - ((2 \cdot FA)/(F1A + F2A))^{-2})/4 \)

16 \( AB = (1 - ((2 \cdot FB)/(F1B + F2B))^{-2})/4 \)

17 \( BA = 1.71 \cdot AA + (1.36 \cdot 36 \cdot RA + B)/(2 \cdot (RA - 1)) \)

18 \( BB = 1.71 \cdot AB + (1.36 \cdot 36 \cdot RB + B)/(2 \cdot (RB - 1)) \)

19 \( C1A = (4 \cdot AA^{-2})/T \)

20 \( C2A = (-4 \cdot AA \cdot BA)/T \)

21 \( C3A = 4 \cdot AA^{-2} \)

22 \( C4A = -4 \cdot AA \cdot BA \)

23 \( C5A = 4 \cdot BA^{-2} \)

24 \( C6A = 4 \cdot (AA/T)^{-2} \)

25 \( C1B = (4 \cdot AB^{-2})/T \)

26 \( C2B = -4 \cdot AB \cdot BB/T \)

27 \( C3B = 4 \cdot AB^{-2} \)

28 \( C4B = -4 \cdot AB \cdot BB \)

29 \( C5B = 4 \cdot BB^{-2} \)

30 \( C6B = 4 \cdot (AB/T)^{-2} \)

31 \( VT = (E1B - E2B - (C1B/C1A) \cdot (E1A - E2A))/(C2B - C1B \cdot C2A/C1A) \)

32 \( UV = (E1A - E2A)/(C1A \cdot C2A \cdot VT/C1A) \)

33 \( V = E4B/C6B \)
34 \( \text{EA} = \text{E1A} - \text{C6A}^{*} \times \text{V/4} - \text{C1A}^{*} \times \text{UV/2} - \text{C2A}^{*} \times \text{VT/2} \cdot \)

35 \( \text{C} = (\text{C4A}/2.) \times (\text{C3A} \times \text{C4B} - \text{C3B} \times \text{C4A}) \)

36 \( \text{D} = (\text{C3A}/4.) \times (\text{C4A} \times \text{C5B} - \text{C5A} \times \text{C4B}) - \text{C5A}^{*} \times (\text{C3A} \times \text{C5B} - \text{C3B} \times \text{C5A}) + \text{C} \)

37 \( \text{E} = \text{EA}^{*} \times (\text{C4A} \times \text{C5B} - \text{C4B} \times \text{C5A}) \)

38 \( \text{D1} = \text{E} - \text{C5A}^{*} \times (\text{E3A} \times \text{C5B} - \text{E3B} \times \text{C5A}) + (\text{C4A}/2.) \times (\text{E3A} \times \text{C4B} - \text{E3B} \times \text{C4A}) \)

39 \( \text{F} = (\text{C3A}/4.) \times (\text{E3A} \times \text{C5B} - \text{E3B} \times \text{C5A}) \)

40 \( \text{D2} = \text{F} - \text{EA}^{*} \times (\text{C3A} \times \text{C5B} - \text{C3B} \times \text{C5A}) + (\text{C4A}/2.) \times (\text{C3A} \times \text{E3B} - \text{C5B} - \text{E3A}) \)

41 \( \text{G} = (\text{C3A}/4.) \times (\text{C4A} \times \text{E3B} - \text{C4B} \times \text{E3A}) \)

42 \( \text{D3} = \text{G} - \text{C5A}^{*} \times (\text{C3A} \times \text{E3B} - \text{C3B} \times \text{E3A}) + \text{EA}^{*} \times (\text{C3A} \times \text{C4B} - \text{C3B} \times \text{C4A}) \)

43 \( \text{U} = \text{D1}/\text{D} \)

44 \( \text{TS} = \text{D2}/\text{D} \)

45 \( \text{UT} = \text{D3}/\text{D} \)

210 PUNCH 110,X,Y,UV,VT

211 PUNCH 111,UT,TS,U,V

GO TO 200

STOP

END