A Study of the Properties of Parametric Programming

Arjun Kumar Batr

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A STUDY OF THE PROPERTIES OF
PARAMETRIC PROGRAMMING

BY

ARJUN KUMAR BATRA

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
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1971
A STUDY OF THE PROPERTIES OF
PARAMETRIC PROGRAMMING

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser

Date

Head, Mechanical Engineering

Department

Date
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AKB
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CHAPTER I

INTRODUCTION

Programming is defined as the planning of activities for the sake of optimization (2). When linear constraints are assumed, together with a linear objective function, the optimization is defined as solving a linear programming problem (6).

Linear programming was originally developed by Dantzig, Wood and others for the U. S. Air Force (3). Since it was first published in 1947, it has been used widely in industrial as well as military situations. Examples of specific exotic and straight forward applications of the linear programming technique can be found in textbooks, monographs and technical papers in the several fields in which it has been used (3,5,6,11,12,14).

The simplex procedure is the most powerful and efficient technique in existence for the solution of linear programming problems. It is assumed that the reader is familiar with the basic simplex procedure and nomenclature. Within the generalized simplex method, there are several specialized techniques that are superior for specific problem areas. It is desirable to review some of these techniques.

Integer Programming

Integer programming involves linear programming problems in which the solution values must all be integers. Obviously any linear programming solution can be reduced to integer values, but this will not necessarily be an optimum solution.
The primary solution procedure utilizes the concept of "cutting planes". To describe this briefly, it is necessary to introduce a few more concepts, the first being the function known as an integral part.

The integral part of a real number 'a' is denoted by \([a]\) and is the greatest integer which satisfies

\[
[a] \leq a
\]

The fractional part of a real number 'a' is the quantity

\[
\{a - [a]\}.
\]

If this difference is denoted by 't', the fractional part has the range

\[
0 \leq t < 1.
\]

Further, if one of the given inequalities is

\[
a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b
\]

then the cutting plane belonging to this inequality is the inequality

\[
t_1x_1 + t_2x_2 + \ldots + t_nx_n \leq t
\]

in which the non-negative numbers \(t_1, t_2, \ldots, t_n\), t are the fractional parts of the coefficients \(a_1, a_2, \ldots, a_n\), b.

This cutting plane inequality replaces the related inequality for the determination of the next optimum solution. If this new
optimum solution is not completely integral, this step is repeated and
an optimal solution in integers is obtained in this way in a finite
number of steps.

This theory was first developed by Gomory (8) in 1958 and has al­
so been used to construct an optimal tour for a salesman visiting
different cities (14).

Many other approaches to integer programming have been considered
and all have been applied successfully to the solution of real problems
but none has been found to be reliable enough to give the optimum so­
lution in a reasonable length of time even if the number of variables
is restricted to fifty (1).

Quadratic Programming

Quadratic programming is the technique of finding a minimum or
maximum solution when the objective function is a quadratic relation
subject to linear inequality constraints. Mathematically this is the
first extension beyond the purely linear programming concept.

One of the techniques suggested for solving such problems makes
use of Kuhn-Tucker conditions and Lagrange multipliers. It modifies
the problem to a linear form for further computation. A better and
simpler approach has been suggested by Beale (1). His approach in­
volves partitioning the variables into basic and non-basic variables
at each stage and writing the objective function in terms of the non­
basic variables. This procedure is repeated until no improvements
in the objective function can be obtained by increasing one of the non-
basic variables. Convergence is obtained in a finite number of steps.

Sensitivity Analysis and Parametric Programming

Sometimes when a linear programming problem is formulated, the
values given to the coefficients in the objective function, the con-
straint constants and the input-out coefficients are only estimates.
It is important to study the effects of their variation because any
variation in their values may affect the optimum solution.

Sensitivity analysis is the technique that determines the range
of values of the coefficients and constraints in an optimum solution of
a linear programming problem without changing the optimum solution.

Sensitivity analysis suggests that a small change in the value of
one coefficient in objective function shall change the relationship
between the objective function and the constraint equations. At some
value, the change shall be enough to create a different optimal solu-
tion. This is true for other coefficients and constraints also.

When the coefficients in the objective function or the constraint
column are expressed in terms of a variable parameter, and the optimal
solutions are found for the complete range of that parameter, the tech-
nique is called parametric programming.

The area of sensitivity analysis and parametric programming ap-
ppears to be less developed than the other advanced techniques. Al-
though several separate theorems and procedures exist, they have not
been unified. It is therefore proposed that this area be exposed in
deepth and the various procedures unified.
CHAPTER II

PARAMETRIC PROGRAMMING AND SENSITIVITY ANALYSIS
WITH ONE PARAMETER IN OBJECTIVE FUNCTION

The formulation of a linear programming problem sometimes involves the estimation of the objective function coefficients, constraint constants and matrix coefficients. When these coefficients are estimated, it is important to know the ranges of these values for which the solution is still optimum. The investigation of these ranges is called sensitivity analysis.

If sensitivity analysis investigates the range of values on which the coefficients in the problem may vary without changing the optimal solution, then parametric programming may be thought of as investigating the total range of coefficients, which will obviously then provide the ranges for a variety of optimal solutions.

Parametric programming and sensitivity analysis will be discussed in terms of the procedures for determining their values.

Parametric Programming

Parametric programming investigates those situations in which one or more parameters replace the coefficients in the objective function or the constraint column. In single parameter problems, \( \lambda \) will be defined by Manne [13]: "The parameter \( (\lambda) \) is a coefficient that is held constant while performing one part of an analysis, but that is considered to be a variable for purposes of analysis as a whole." The
theoretical range of interest in the values of $\lambda$ is along the entire line of real numbers, that is, $-\infty \leq \lambda \leq +\infty$.

In some special cases $\lambda$ may be defined over a more restricted range. For example Gass (7) defined $\lambda$ as the ratio of the cost of a unit increase in production to the cost of storing a unit for one period of time in a production scheduling program. In this case, the range of interest is $0$ to $+\infty$ since negative values of $\lambda$ do not have meaning.

It may be hypothesized that $\lambda$ has a range of values for every basic feasible solution. However deeper consideration suggests that for any given objective function, there must be a point that is the antithesis of the objective function. That is, if the objection function is to maximize, there is always a basic feasible solution that is minimal and vice versa. This will be demonstrated later in the chapter.

The ranges of values of $\lambda$ for each solution that make that solution optimal are described as the characteristic intervals for that solution. Therefore, each optimal solution has associated with it a characteristic interval. Lambda may have both point and finite solutions. These characteristic intervals form a connected set which is the real line. Once an optimum solution has been found for any value of $\lambda$, parametric programming can be used to determine the end points of the interval. From each end point, the neighboring characteristic interval can be found in a similar manner until the complete real line has been found. All such values of $\lambda$ can be associated with their optimum solutions. All ranges of $\lambda$ can be determined in a finite number of iterations,
usually between two to three times the range of the matrix (6).

The procedure for finding the values for a parameter in the objective function shall be derived first. Once this has been found, the procedure for finding values for parameters in the constraint column follows logically.

Derivation of Parametric Relations

A linear parametric problem can be stated as follows:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{j=1}^{n} (c_j + \lambda c_j') x_j \\
\text{Subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \\
& \quad x \geq 0 (i = 1,2,...,m) \\
& \quad (j = 1,2,...,n)
\end{align*}
\]

[2.1]

where

- \( c \) is the coefficient of the jth variable in objective function.
- \( \lambda c_j' \) is the parametric change in the value of \( c_j \).
- \( a_{ij} \) is a coefficient in the simplex matrix in ith row and jth column.
- \( b_i \) is a constraint constant for the ith equation.
- \( x_j \) is the jth variable.
It is assumed that the problem is nondegenerate and that a basic feasible solution is available. The parameter $\lambda$ is bounded by limits $\xi$ and $\phi$ where $\xi$ and $\phi$ may be either finite or infinite. If they are finite then $\xi$ for one solution is equal to $\phi$ for the next. If they are infinite, then no finite limit exists for that objective function.

It is obvious that the presence of $\lambda$ in the objective function will result in $\lambda$ appearing in the $(z_j - c_j)$ row. The $(z_j - c_j)$ row can be expressed as $(\alpha_j + \lambda \rho_j)^*$ and for a minimizing function, the solution is optimum only if

$$\alpha_j + \lambda \rho_j \leq 0 \text{ for all } j$$

[2.2]

$\alpha_j$ and $\lambda \rho_j$ can either be zero, positive or negative in [2.2].

There are six possibilities.

1. $\alpha_j$ is negative; $\lambda \rho_j$ is zero.
2. $\alpha_j$ is zero; $\lambda \rho_j$ is negative.
3. $\alpha_j$ is zero; $\lambda \rho_j$ is zero.
4. $\alpha_j$ is negative; $\lambda \rho_j$ is positive.
5. $\alpha_j$ is positive; $\lambda \rho_j$ is negative.
6. $\alpha_j$ is negative; $\lambda \rho_j$ is negative.

Each possibility must be discussed separately.

Case # 1.

If $\alpha_j$ is negative and $\lambda \rho_j$ is zero, then $\lambda$ does not appear in that column in the $(z_j - c_j)$ row. Further if all $\rho_j$ are zero, then the

*An initial tableau with $\alpha$ and $\rho$ values in the $(z_j-c_j)$ row is found on page 13.
parameter \( \lambda \) does not affect the problem at all and the problem is not capable of parametric analysis.

Case # 2.

If \( \alpha_j \) is zero and \( \lambda \beta_j \) is negative, then \( \lambda \) can vary from \( + \lambda \beta_j \) to \( -\infty \), without changing the optimum solution.

Case # 3.

\( \alpha_j = \lambda \beta_j = 0 \) is true for those \( j \) for which the corresponding variable is in solution.

Case # 4.

If \( \alpha_j \) is negative and \( \lambda \beta_j \) is positive, then

\[
\lambda \beta_j \text{ can vary from } + \alpha_j \text{ to } -\infty.
\]

Equation [2.2] can be stated as

\[
- [\alpha_j] + \lambda \beta_j \leq 0
\]

or

\[
\lambda \leq \frac{[\alpha_j]}{\beta_j} \text{ for } \alpha_j < 0
\]

\[
\beta_j > 0
\]

[2.3]

It is obvious from [2.3] that as \( \beta_j \) approaches zero, \( \lambda \) approaches \( +\infty \) and hence is unbounded in that direction. Therefore if no \( \beta_j > 0 \) and \( \alpha_j \) is negative, the value of \( \lambda \) would be unbounded.
Case # 5.

If $\alpha_j$ is positive and $\lambda \beta_j$ is negative, then

$$\lambda \beta_j \text{ can vary from } -\alpha_j \text{ to } -\infty.$$  

Equation [2.2] can be stated as

$$\alpha_j - \lambda |\beta_j| \leq 0$$  

or

$$\lambda \leq \frac{\alpha_j}{|\beta_j|}$$

or

$$\lambda \geq - \frac{\alpha_j}{|\beta_j|} \text{ for } \alpha_j > 0 \quad \beta_j < 0$$

[2.4]

It is obvious from [2.4] that as $|\beta_j|$ approaches zero, $\lambda$ approaches minus infinity. Therefore if no $\beta_j < 0$ and $\alpha_j$ is positive, the value of $\lambda$ is unbounded in that direction.

Case # 6.

If $\beta_j$ is negative and $\lambda_j$ is negative, then Case # 5 is a special case of Case # 6, in that when $\alpha_j$ becomes negative, Case # 5 and Case # 6 are identical.
Since it is assumed that \( \lambda \) is bounded by \( \delta \) and \( \varnothing \), relations \([2.3]\) and \([2.4]\) can be written as

\[
\delta = \left( -\frac{\alpha_j}{|\beta_j|} \right) \text{ for } \beta_j < 0 \quad \alpha_j > 0
\]

\[
\delta = -\infty \text{ if all } \beta_j > 0
\]

and

\[
\varnothing = \left( \frac{|\alpha_j|}{\beta_j} \right) \text{ for } \beta_j > 0 \quad \alpha_j < 0
\]

\[
\varnothing = +\infty \text{ if all } \beta_j < 0
\]

Relations \([2.5]\) and \([2.6]\) provide a multiplicity of ranges and in them, there is only one correct range. It is obvious that if limits other than the most restrictive ones are used, the initial condition \((\alpha_j + \lambda \beta_j) \leq 0\) is violated for at least one element in the \((z_j - c_j)\) row. Therefore the correct limits are the most restrictive limits; or

\[
\max \left( -\frac{\alpha_j}{|\beta_j|} \right) \leq \lambda \leq \min \left( \frac{|\alpha_j|}{\beta_j} \right)
\]

\[
\beta_j < 0 \quad \beta_j > 0
\]

\[
\alpha_j > 0 \quad \alpha_j < 0
\]

\[
\text{or}
\]

\[
-\infty \text{ if all } \beta_j > 0 \quad +\infty \text{ if all } \beta_j < 0
\]
A new solution for \( \lambda = \delta \) can be obtained by introducing that variable in solution for which \((\alpha_j + \delta \beta_j) = 0\). Thus the lower limit for one solution is the upper limit for the next interval and a new lower limit can be found. In the same way, by substituting for the solution at the upper limit, the current upper limit becomes lower limit for the next interval and a new upper limit may be determined. Thus all possible values of \( \lambda \) can be found from the basic feasible solutions that satisfy the objective function.

If, in the process of finding a new feasible solution for \( \lambda = \delta \), the variable chosen to go into solution satisfies \( \alpha_j + \delta \beta_j = 0 \) but cannot enter the basis because all the column elements \((a_{ij})\) in the simplex matrix are less than or equal to zero, there will be no finite minimum solution for \( \lambda = \delta \). The parameter \( \lambda \) shall, in that case, lie between

\[-\infty \leq \lambda \leq \delta.\]

Similar argument can be applied for a new solution for \( \lambda = \phi \), and by the same reasoning

\[0 \leq \lambda \leq +\infty.\]

If the objective function is a maximizing function, then \( \lambda \), by the same procedure, has a range of variation given by

\[
\max \left( \frac{|\alpha_j|}{\rho_j} \right) \leq \lambda \leq \min \left( \frac{\alpha_j}{|\beta_j|} \right)
\]

or

- \(-\infty\) if all \( \rho_j \leq 0 \)

or

- \(+\infty\) if all \( \beta_j > 0 \)


\[-[2.8]\]
The single parameter programming concept is demonstrated by solving an example for maximizing as well as minimizing objective functions.

Given the following linear programming problem:

Maximize \( Z = (2 + \lambda)x_1 + (3 - \lambda)x_2 - \lambda \)

Subject to
\[
\begin{align*}
-x_1 + 2x_2 &\leq 4 \\
x_1 + x_2 &\leq 5 \\
2x_1 - x_2 &\leq 8 \\
x_j &\geq 0
\end{align*}
\]

By the normal procedure, the initial tableau is established:

**Initial Tableau**

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>(-\lambda)</th>
<th>((2+\lambda)(3-\lambda))</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Objective Column</strong></td>
<td><strong>Basis</strong></td>
<td><strong>P_0</strong></td>
<td><strong>P_1</strong></td>
<td><strong>P_2</strong></td>
<td><strong>P_3</strong></td>
</tr>
<tr>
<td>0</td>
<td>P_3</td>
<td>4</td>
<td>-1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>P_4</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>P_5</td>
<td>8</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
(z_j - c_j)\begin{cases} 
(m+1) & 0 & -2 & -3 & 0 & 0 & 0 \\
(m+2) & +\lambda & -\lambda & +\lambda & 0 & 0 & 0
\end{cases}
\]
This is a basic feasible solution. Nevertheless, applying the
parametric programming formula does not give a characteristic interval
and there are no values of $\lambda$ for the solution $x_1=0$, $x_2=0$ for this
maximizing function.

Pivoting on vector $P_1$ gives Tableau 2:

**Tableau 2**

<table>
<thead>
<tr>
<th>Objective Column</th>
<th>Objective Row</th>
<th>Basis</th>
<th>$-\lambda$</th>
<th>$(2+\lambda)$</th>
<th>$(3-\lambda)$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$P_3$</td>
<td>8</td>
<td>0</td>
<td>$3/2$</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$P_4$</td>
<td>1</td>
<td>0</td>
<td>$3/2$</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td></td>
</tr>
<tr>
<td>$(2+\lambda)$</td>
<td>$(m+1)$</td>
<td>$P_1$</td>
<td>4</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>$(m+2)$</td>
<td>$(m+2)$</td>
<td></td>
<td>8</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Applying parametric programming formula for finding the characteristic intervals of $\lambda$ for this solution,

$$\max\left(-\frac{|\alpha_j|}{\beta_j}\right) \leq \lambda \leq \min\left(-\frac{\alpha_j}{|\beta_j|}\right)$$

- $\beta_j > 0$
- $\beta_j < 0$
- $\alpha_j < 0$
- $\alpha_j > 0$

or

- $-\infty$ if all $\beta_j \leq 0$ + $\infty$ if all $\beta_j \geq 0$
or

\[
\max \left( -\frac{4}{1/2} ; -\frac{1}{1/2} \right) \leq \lambda \leq +\infty
\]

or

\[8 \leq \lambda \leq +\infty\]

Because the \(P_2\) vector comprises the lower limit of the characteristic range, this vector is inserted into the solution giving Tableau 3.

### Tableau 3

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>(-\lambda)</th>
<th>((2+\lambda))</th>
<th>((3-\lambda))</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective Column</td>
<td>Basis</td>
<td>(P_0)</td>
<td>(P_1)</td>
<td>(P_2)</td>
<td>(P_3)</td>
<td>(P_4)</td>
<td>(P_5)</td>
</tr>
<tr>
<td>0</td>
<td>(P_3)</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>((3-\lambda))</td>
<td>(P_2)</td>
<td>2/3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>((2+\lambda))</td>
<td>(P_1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>((m+1))</td>
<td></td>
<td>32/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>((m+2))</td>
<td></td>
<td>14/3\lambda</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-\lambda/3</td>
<td>2/3\lambda</td>
</tr>
</tbody>
</table>

or thus \[1/2 \leq \lambda \leq 8\]

In a similar manner, the remaining intervals can be determined and are shown in Table 2-1.
### TABLE 2-1

**OUTPUT TABLE FOR A MAXIMIZING FUNCTION**

<table>
<thead>
<tr>
<th>Values of real Variables</th>
<th>Objective function value</th>
<th>Characteristic Intervals For λ</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( Z(\lambda) )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>4.0</td>
<td>0</td>
<td>( 8 + 5\lambda )</td>
</tr>
<tr>
<td>4.33</td>
<td>0.67</td>
<td>( 32/3 + 14/3\lambda )</td>
</tr>
<tr>
<td>2.0</td>
<td>3.0</td>
<td>13</td>
</tr>
<tr>
<td>0</td>
<td>2.0</td>
<td>13</td>
</tr>
<tr>
<td>0</td>
<td>2.0</td>
<td>( 6 - \lambda )</td>
</tr>
</tbody>
</table>

If the same problem is treated as a minimization problem by taking the negative of the objective function, a similar set of solutions can be found as shown in Table 2-2.

### TABLE 2-2

**OUTPUT TABLE FOR A MINIMIZING FUNCTION**

<table>
<thead>
<tr>
<th>Values of real Variables</th>
<th>Objective function value</th>
<th>Characteristic Intervals For λ</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( Z(\lambda) )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0</td>
<td>2.0</td>
<td>( 6 - \lambda )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( 3 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>4.0</td>
<td>0</td>
<td>( 8 + 5\lambda )</td>
</tr>
<tr>
<td>0</td>
<td>2.0</td>
<td>( 6 - \lambda )</td>
</tr>
</tbody>
</table>
Figure 2-1. Graph showing variation of the value of the parameter for a maximizing function.
Figure 2-2. Graph showing variation of the value of the parameter for a minimizing function.
The characteristic ranges for the two problems can also be shown graphically. In Figure 2-1, the limiting objective functions for the various maximizing solutions are shown. The arrows indicate the direction of the driving force. In Figure 2-2, the minimizing function is shown the same way.

In Figure 2-1, it can be noted that no value of $\lambda$ will provide an optimal solution for the basic feasible solution of $x_1 = 0, x_2 = 0$. In Figure 2-2, there is no optimal solution for extreme points (2,3) and (4,3.3, 0.67). These, of course, are the antithesis points that were suggested at the beginning of the chapter and can be recognized by the inability to find a real characteristic interval.

Sensitivity Analysis

Sensitivity analysis is the determination of the range over which the coefficients and constraints can vary before a solution changes. Within the sensitivity range of a given coefficient or constraint, the optimum value of performance may change but the variables which are not in the solution do not enter the solution.

From this definition, it is obvious that sensitivity analysis is parametric programming over a single range of values for $\lambda$. In parametric programming, the range is found for each solution that will make it optimal. In sensitivity analysis, the optimal solution is found first and the range of values computed to not change that optimality.
Gass and Saaty (7) found the sensitivity range as

\[
\max \left( \frac{z_j - c_j}{|a_{kj}|} \right) \leq \Delta c \leq \min \left( -\frac{z_j - c_j}{a_{kj}} \right)
\]

\[
a_{kj} < 0 \quad \text{or} \quad a_{kj} > 0
\]

or

\[
-\infty \text{ if all } a_{kj} \geq 0 \quad \text{or} \quad +\infty \text{ if all } a_{kj} \leq 0
\]

[2.9]

It is only necessary to relate their terminology to that used in parametric programming to satisfy the unifying concept.

\( \Delta c_j \) is defined as the change in the value of the \( j \)th coefficient in the objective function. In equation [2.1], this change was called \( \lambda \). Further in equation [2.1], \( (z_j - c_j) \) was described as \( (\alpha_j + \lambda \beta_j) \) by separation of the constant. Since \( z_j = \sum c_j a_{ij} \), for \( P_j \) in the solution, \( \Delta c_j(\lambda) \) will appear in \( z_k \) only. If \( P_j \) is not in the solution, \( \Delta c_j(\lambda) \) will occur in \( c_j \) only. Each possibility is considered separately.

If \( P_j \) is in the solution, then for the \( k \)th column

\[
z_k = \Delta c_j x a_{jk} + \sum c_i a_{ik}
\]

and

\[
z_k - c_k = \Delta c_j x a_{jk} + \sum c_i a_{ik} - c_k
\]

or

\[
\Delta c_j = \frac{\sum c_i a_{ik} - c_k}{-a_{jk}}
\]
but \((c_i - a_{ik} - c_k)\) is the \(\alpha\) term and \(a_{jk}\) the \(\beta\) term. Also the values of \(a_{jk}\) may be either positive or negative in \(j\)th row of the simplex matrix of the optimum solution, thereby giving

\[
\lambda = \Delta c_j = \frac{\alpha_j}{\beta_j}
\]

for either limit.

Gass and Saaty do not consider the case where \(P_j\) is not in the solution. However

\[
z_j - c_j = z_j - (c_j + \Delta c_j)
\]

where \((z_j - c_j)\) is now the \(\alpha\) term and \(\Delta c_j\) is \(\lambda\). The critical limit is when

\[
(z_j - c_j) = \Delta c_j
\]

or

\[
\alpha = \lambda
\]

that is, \(\beta\) has the value of unity.

**SENSITIVITY ANALYSIS ON THE ANALOG COMPUTER**

Analog Computers have been programmed for solving linear programming problems. They provide an efficient tool for the evaluation of parametric changes in a linear programming model. Once a solution has been obtained on an Analog Computer, further solutions can be obtained quickly for different values of the parameters [3,9].
The Analog Computer consists of a number of electronic units called operational amplifiers used in conjunction with simple resistor-capacitor circuits. In addition, specialized devices like diodes, potentiometers, relays, function generators are used as auxiliary equipment. The Analog Computer establishes the relations through analogy of the mathematical relations and the circuits.

The parameters in a linear programming problem are represented by potentiometer settings on a Analog Computer. Any variation in parameters can be accomplished by the manipulation of the appropriate potentiometer. The ability to manipulate the parameters of a problem implies that the Analog Computer is ideally suited for sensitivity analysis and parametric programming.

In linear programming on an Analog Computer the size of the problem that can be solved is restricted by the number of computing amplifiers available and the number of inputs to each amplifier. The number of required amplifiers is bowed between \((m + 2n)\) and \((2m + 3n)\) where \(m\) equals the number of constraint equations and \(n\) equals the number of variables. Analog Computers work well on small problems. The limitation on amplifiers prevents their use in large problems.


Because of the interest in parametric programming, IBM has included procedures in its macrolanguage program LPS-360 to provide this form of solution.
LPS-360 has subroutines for sensitivity analysis and for solving parametric programming problems, with one parameter in the objective function. LPS-360 system can process problems of 1500 equations and an arbitrary large number of variables on the 64k version computer (8).

Parametric programming problems with a single parameter in the objective function and sensitivity analysis of the cost coefficients of a linear programming problem have been derived. Sensitivity analysis has been presented as a special case of parametric programming in which the parameter \( \lambda \) replaces only one cost coefficient in the objective function.

In the following chapter, parametric programming when the parameter is located in the constraint constants will be derived and discussed.

Finally, parametric programming where there are two or more parameters in the objective function will be derived and a generalized approach for a two parameter problem shall also be presented.
CHAPTER III

PARAMETRIC PROGRAMMING AND SENSITIVITY ANALYSIS ON CONSTRAINTS

A linear programming problem with a parameter in the constraint constant can basically be solved in the same manner as a parametric problem with a single parameter in the objective function.

To apply the simplex computational procedure to a linear programming problem, the problem is assumed to be feasible and nondegenerate for every basic feasible solution. For a linear programming problem to stay feasible, it is necessary that the constraint constants are either zero or positive.

It will be remembered that the constraint column \( b_i \) is used in the determination of the variable to leave the basis. That is, for a minimizing problem, if the condition \( (z_j - c_j) > 0 \) holds and if there is at least one \( a_{ij} > 0 \) for \( i = 1, 2, \ldots, m \), then

\[
\theta = \frac{b_i}{a_{ij}} > 0 \text{ for } a_{ij} > 0, \text{ and}
\]

that variable is eliminated from the basis for which \( \theta \) is minimum. The new basis can be manipulated similarly to the previous one. This process continues either until all \( (z_j - c_j) \leq 0 \) or until for some \( (z_j - c_j) \geq 0 \), all \( a_{ij} \leq 0 \).
In general, a linear parametric programming problem with the parameter in the constraint column can be stated as

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{n} c_j \cdot x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} \cdot x_j = b_i + \lambda b'_i \\
& \quad (i = 1, 2, \ldots, m) \\
& \quad (j = 1, 2, \ldots, n) \\
& \quad x_j \geq 0
\end{align*}
\]

where

\[b_i\] is the constraint constant for the ith equation.

\[\lambda b'_i\] is the parametric change in the value of \(b_i\).

All other variables are as previously defined.

It is assumed that the parameter \(\lambda\) is bounded by finite limits \(\delta\) and \(\phi\) except for the final characteristic regions which may not be bounded. Once an optimum solution has been obtained for a range of values of \(\lambda\), the selection of next variable to be introduced into the basis and the variable to be eliminated that would keep the constant column of the transformed equations non-negative and maintain the optimality of the solution is determined by formula proved by Gass (6).
in the case where variable $x_1$ corresponding to $\lambda = 6$ or $\lambda$ is eliminated from the basis and variable $x_k$ is introduced into the basis, then the new solution is minimum for at least one value of $\lambda$.

It is obvious that for an optimum solution with the values of the variables in the basis $X = (a_{10}, a_{20}, \ldots, a_{m0})$ to stay feasible, the constant column must be non-negative, that is for the problem when $\lambda = \emptyset$,

$$a_{i0} = \alpha_i + \lambda \beta_i \geq 0 \text{ for all } i$$  \[3.2\]

[3.2] can be expressed as

$$a_{i0} = \alpha_i + \lambda \beta_i \geq 0 \text{ for all } i$$  \[3.3\]

$\alpha_i$ and $\lambda \beta_i$ can either be zero, positive or negative. There are six possibilities.

1. $\alpha_i$ is positive; $\lambda \beta_i$ is zero.
2. $\alpha_i$ is zero; $\lambda \beta_i$ is positive.
3. $\alpha_i$ is zero; $\lambda \beta_i$ is zero.
4. $\alpha_i$ is positive; $\lambda \beta_i$ is negative.
5. $\alpha_i$ is positive; $\lambda \beta_i$ is positive.
6. $\alpha_i$ is negative; $\lambda \beta_i$ is positive.

Each possibility must be discussed separately.
Case # 1

If $\alpha_i$ is positive and $\lambda \beta_i$ is zero, then $\lambda$ does not appear in the constant column. Further if all $\beta_i$ are zero, the parameter $\lambda$ does not affect the problem and the problem is not capable of parametric analysis.

Case # 2

If $\alpha_i$ is zero and $\lambda \beta_i$ is positive, then $\lambda$ can vary from 0 to $+\infty$.

Case # 3

If $\alpha_i = 0$ and $\beta_i = 0$, is true for one or more elements in the constraint column, the problem is degenerate. Degeneration violates the basic assumption of non-degeneration.

Case # 4

If $\alpha_i$ is positive and $\lambda \beta_i$ is negative, then, $\alpha_i$ can vary from $+ \lambda \beta_i$ to $+\infty$ and equation [3.3] can be written

$$\alpha_i - \lambda |\beta_i| \geq 0$$

or

$$\lambda \geq \frac{\alpha_i}{|\beta_i|} \quad \text{for } \alpha_i > 0$$

$$\beta_i < 0$$

[3.4]

It is obvious from [3.4] that as $|\beta_i|$ approaches zero, $\lambda$ approaches $+\infty$ and hence is unbounded in that direction. Therefore if no $\beta_i < 0$, the value of $\lambda$ would be unbounded.
Case # 5

If $\alpha_1$ is positive and $\lambda \beta_1$ is positive, then equation [3.3] remains

$$\alpha_1 + \lambda \beta_1 \geq 0$$

or

$$\lambda \geq -\frac{\alpha_1}{\beta_1} \text{ for } \alpha_1 > 0$$

[3.5]

$$\beta_1 > 0$$

It is noted from [3.5] that as $\beta_1$ approaches zero, $\lambda$ approaches minus infinity. Therefore if no $\beta_1 > 0$, value of $\lambda$ is unbounded in that direction.

Case # 6

If $\alpha_1$ is negative and $\beta_1$ is positive, then $\alpha_1$ can vary from $-\lambda \beta_1$ to $+\infty$. Case # 6 is a special case of Case # 5, in that as $\alpha_1$ becomes more positive, Case # 5 and Case # 6 are identical.

Since it is assumed that $\lambda$ is bounded by $\delta$ and $\phi$, relations [3.4] and [3.5] can be expressed as

$$\phi = \left( \frac{\alpha_1}{|\beta_1|} \right) \quad \text{for } \alpha_1 > 0$$

$$\beta_1 < 0$$

$$= +\infty \text{ if all } \beta_1 \geq 0$$

[3.6]
and

\[ \delta = \left( - \frac{\alpha_i}{\rho_i} \right) \quad \text{for} \quad \alpha_i > 0 \]

\[ \rho_i > 0 \]

\[ = -\infty \quad \text{if all} \quad \rho_i \leq 0 \]

Relations [3.6] and [3.7] provide a multiplicity of ranges of \( \lambda \) and in them there is only one correct range. It is obvious that if limits other than most restrictive are used, the initial condition \((\alpha_1 + \lambda \rho_1) \geq 0\) is violated for at least one case in the constant column. Therefore the correct limits are the most restrictive of the limits, or

\[
\max \left( - \frac{\alpha_i}{\rho_i} \right) \leq \lambda \leq \min \left( \frac{\alpha_i}{|\rho_i|} \right)
\]

\[
\rho_i > 0 \quad \alpha_i > 0 \quad \text{or} \quad -\infty \quad \text{if all} \quad \rho_i \leq 0
\]

\[
\rho_i < 0 \quad \alpha_i > 0 \quad \text{or} \quad +\infty \quad \text{if all} \quad \rho_i \geq 0
\]

Relation [3.8] is also applicable to a maximizing function. The computation differs during optimizing procedure because, in maximizing, a solution is optimum only when all \((z_j - c_j) \geq 0\). The feasibility condition, that is, \((\alpha_1 + \lambda \rho_1) \geq 0\) is the same in both cases.

Also, as \( \lambda \) is increased, the solution may remain optimum but it may not stay feasible.
An Example

Maximize \( x = x_1 + x_2 \).

Subject to

\[-x_1 + 2x_2 \leq 10\]
\[2x_1 + x_2 \leq 30\]
\[x_1 + x_2 \leq 24 - 2\lambda\]
\[x_j \geq 0\]

The initial tableau can be established by the normal procedure.

<table>
<thead>
<tr>
<th>Objective Column</th>
<th>Basis</th>
<th>Initial Tableau</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Objective Row</td>
<td>0</td>
</tr>
<tr>
<td>Objective Column</td>
<td></td>
<td>P_0  P_1  P_2  P_3  P_4  P_5</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>P_3</td>
<td>10   -1   2   1   0   0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>P_4</td>
<td>30   2    1   0   1   0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>P_5</td>
<td>(24 - 2\lambda) 1  1   0   0   1</td>
<td></td>
</tr>
</tbody>
</table>

\((z_j - c_j)\) 0 -1 -1 0 0 0 0

Pivoting on column P_1 gives 2nd tableau.

<table>
<thead>
<tr>
<th>Objective Column</th>
<th>Basis</th>
<th>2nd Tableau</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Objective Row</td>
<td>0</td>
</tr>
<tr>
<td>Objective Column</td>
<td></td>
<td>P_0  P_1  P_2  P_3  P_4  P_5</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>P_3</td>
<td>25   0    5/2 1   1/2 0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>P_1</td>
<td>15   1    1/2 0   1/2 0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>P_5</td>
<td>(9 - 2\lambda) 0  1/2 0   -1/2 1</td>
<td></td>
</tr>
</tbody>
</table>

15 0 -1/2 0 1/2 0
Pivoting on column \( P_2 \) gives 3rd tableau.

<table>
<thead>
<tr>
<th>Objective Column</th>
<th>Basis</th>
<th>( P_0 )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
<th>( P_5 )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( P_2 )</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>2/5</td>
<td>1/5</td>
<td>0</td>
<td>This is an optimum solution.</td>
</tr>
<tr>
<td>1</td>
<td>( P_1 )</td>
<td>10</td>
<td>1</td>
<td>0</td>
<td>-1/5</td>
<td>2/5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( P_5 )</td>
<td>(4 - 2( \lambda ))</td>
<td>0</td>
<td>0</td>
<td>-1/5</td>
<td>-3/5</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Third iteration gives an optimum solution. To determine the range of \( \lambda \) for this solution, \((4 - 2\lambda) \geq 0\) or \( \lambda \leq 2\).

To choose the next variable to enter the solution, the following formula is used.

\[
\frac{z_k - c_k}{a_{lk}} = \min_{a_{lk} \neq 0} \frac{z_j - c_j}{a_{lk}}
\]

This formula is applicable only to vector \( P_5 \) in the solution because \( P_5 \) has the corresponding value \((4 - 2\lambda)\).

Therefore,

\[
\frac{z_k - c_k}{a_{lk}} = \min_{a_{lk} < 0} \left( \frac{1/5}{-1/5}, \frac{3/5}{-3/5} \right).
\]

Thus either vector \( P_3 \) or \( P_4 \) can enter the solution and the pivot element shall be the corresponding \( a_{lk} \).
Vector $P_3$ is selected arbitrarily and 4th tableau is obtained.

<table>
<thead>
<tr>
<th>Objective Column</th>
<th>Basis (P_0)</th>
<th>(P_1)</th>
<th>(P_2)</th>
<th>(P_3)</th>
<th>(P_4)</th>
<th>(P_5)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(P_2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>(z_j - c_j)</td>
</tr>
<tr>
<td>1</td>
<td>(P_1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>(24 - 2\lambda)</td>
</tr>
<tr>
<td>0</td>
<td>(P_3)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-5</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore

\[
18 - 4\lambda \geq 0
\]
\[
6 + 2\lambda \geq 0
\]
\[
-20 + 10\lambda \geq 0
\]

Relation [3.8] as given below, when applied to iteration 4 gives the characteristic region for this solution.

\[
\max \left(-\frac{\alpha_i}{\beta_i}\right) \leq \lambda \leq \min \left(\frac{\alpha_i}{\beta_i}\right)
\]

\[\beta_i > 0\] \[\alpha_i > 0\] \[-\infty \text{ if all } \beta_i \leq 0\] \[+\infty \text{ if all } \beta_i \geq 0\]

or

\[
\max \left(-\frac{6}{2}, -\frac{20}{10}\right) \leq \lambda \leq \left(\frac{18}{4}\right)
\]

\[2 \leq \lambda \leq 9/2\]
Pivoting now on column $P_4$ gives 5th tableau.

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>Basis</th>
<th>$P_0$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Column</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$P_4$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td></td>
<td></td>
<td>This is an</td>
</tr>
<tr>
<td>1</td>
<td>$P_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td>optimum</td>
</tr>
<tr>
<td>0</td>
<td>$P$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td>solution</td>
</tr>
<tr>
<td></td>
<td>$(z_j - c_j)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Applying the routine formula for calculating the characteristic range

$$(-\frac{18}{4}) \leq \lambda \leq \text{min} \left(\frac{24}{2}, \frac{34}{2}\right)$$

$$\frac{9}{2} \leq \lambda \leq 12$$

Since there is no $a_{1k} < 0$ in the row corresponding to $P_1$ in the solution, therefore no feasible solution exists for $\lambda > 12$.

In tabulated form,

**TABLE 3-1**

**OUTPUT TABLE OF THE PRIMAL SOLUTION**

<table>
<thead>
<tr>
<th>Values of real Variables</th>
<th>Objective Function Value $Z(\lambda)$</th>
<th>Characteristic Interval For $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>$-\infty \leq \lambda \leq 2$</td>
</tr>
<tr>
<td>$(6 + 2\lambda)(18 - 4\lambda)$</td>
<td>$(24 - 2\lambda)$</td>
<td>$2 \leq \lambda \leq 4.5$</td>
</tr>
<tr>
<td>$(24 - 2\lambda)$ 0</td>
<td>$(24 - 2\lambda)$</td>
<td>$4.5 \leq \lambda \leq 12$</td>
</tr>
</tbody>
</table>

No feasible solution exists $\lambda > 12$
Application of Dual Algorithm to Solve Parametric Model

A linear programming problem in which the parameter is contained in the constraints can also be solved by using the concept of duality and the technique previously developed for solving one parameter objective function problems.

Since the theory of duality is well known and the objective function parameter technique has been discussed, it is adequate to move directly to an example. The same example as used earlier in the chapter shall be solved by this technique.

Maximize \( z = x_1 + x_2 \)
Subject to \( -x_1 + 2x_2 \leq 10 \)
\( 2x_1 + x_2 \leq 30 \)
\( x_1 + x_2 \leq (24 - 2\lambda) \)
\( x_j \geq 0 \)

When \([3.9] \) is put in dual form, the redefined function is to

Minimize \( z = 10z_1 + 30z_2 + (24 - 2\lambda)z_3 \)
Subject to \( -z_1 + 2z_2 + z_3 \geq 1 \)
\( 2z_1 + z_2 + z_3 \geq 1 \)
\( z_j \geq 0 \)

Since this problem is a symmetrical primal dual problem, the condition restricting the variables to be non-negative remains.
Making the necessary changes for maximizing the objective function in [3.10], the inequations and objective function can be written and placed in the initial tableau.

Tableau 1

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>0</th>
<th>-10</th>
<th>-30</th>
<th>-(24 + 2(\lambda))</th>
<th>0</th>
<th>0</th>
<th>-M</th>
<th>-M</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>This is not a feasible</td>
</tr>
<tr>
<td>Column</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>solution</td>
</tr>
<tr>
<td>Basis</td>
<td>P_6</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>-M</td>
<td>P_7</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Pivoting on vector \(P_2\) leads to second tableau and a second pivot on \(P_i\) will give a feasible solution in third tableau.

Tableau 3

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>0</th>
<th>-10</th>
<th>-30</th>
<th>-(24 + 2(\lambda))</th>
<th>0</th>
<th>0</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>This is</td>
</tr>
<tr>
<td>Column</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>a feasible</td>
</tr>
<tr>
<td>Basis</td>
<td>P_2</td>
<td>3/5</td>
<td>1</td>
<td>0</td>
<td>3/5</td>
<td>-2/5</td>
<td>-1/5</td>
</tr>
<tr>
<td>-30</td>
<td>P_1</td>
<td>1/5</td>
<td>0</td>
<td>1</td>
<td>1/5</td>
<td>1/5</td>
<td>-2/5</td>
</tr>
</tbody>
</table>

\((m + 1)\) -20 0 0 +4 +10 +10

\((m + 2)\) 0 0 0 -2\(\lambda\) 0 0
The parametric programming routine finds the characteristic region
for this feasible optimal solution.

\[
\max \left( -\frac{\alpha_i}{\beta_i} \right) \leq \lambda \leq \min \left( \frac{\alpha_i}{\beta_i} \right)
\]

\( \beta_i > 0 \) or \( \alpha_i > 0 \)

or

\( \alpha_i > 0 \) or \( \beta_i < 0 \)

- \( \infty \) if all \( \beta_i \leq 0 \)

+ \( \infty \) if all \( \beta_i \geq 0 \)

or

\( -\infty \leq \lambda \leq \frac{4}{2} \)

Pivoting on \( P_3 \) will give the next range of solutions in tableau 4.

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>0</th>
<th>-10</th>
<th>-30</th>
<th>(24 + 2( \lambda ))</th>
<th>0</th>
<th>0</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td>Basis</td>
<td>( P_0 )</td>
<td>( P_1 )</td>
<td>( P_2 )</td>
<td>( P_3 )</td>
<td>( P_4 )</td>
<td>( P_5 )</td>
</tr>
<tr>
<td>Column</td>
<td>( P_2 )</td>
<td>0</td>
<td>-3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( -(24 - 2\lambda) )</td>
<td>( P_3 )</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( (m + 1) )</td>
<td>-24</td>
<td>-20</td>
<td>0</td>
<td>0</td>
<td>+6</td>
<td>+18</td>
<td></td>
</tr>
<tr>
<td>( (m + 2) )</td>
<td>+2( \lambda )</td>
<td>+10( \lambda )</td>
<td>0</td>
<td>0</td>
<td>+2( \lambda )</td>
<td>-4( \lambda )</td>
<td></td>
</tr>
</tbody>
</table>

Again the parametric programming technique gives the characteristic
region

\[
\max \left( -\frac{20}{10}, -\frac{6}{2} \right) \leq \lambda \leq \left( \frac{18}{4} \right)
\]

\( 2 \leq \lambda \leq 4.5 \)
In the same manner, pivoting on $P_5$ will give a range $4.5 \leq \lambda \leq 12$ and the final pivot on $P_5$ is not possible since all $a_{i5}$ are negative.

In tabulated form,

**TABLE 3-2**

**OUTPUT TABLE OF THE DUAL SOLUTION**

<table>
<thead>
<tr>
<th>Values of Real Variables</th>
<th>Objective Function Value $Z(\lambda)$</th>
<th>Characteristic Interval For $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$(6 + 2\lambda)$</td>
<td>$(18 - 4\lambda)$</td>
<td>$- \infty \leq \lambda \leq + 2$</td>
</tr>
<tr>
<td>$(24 - 2\lambda)$</td>
<td>$(-24 + 2\lambda)$</td>
<td>$2 \leq \lambda \leq 4.5$</td>
</tr>
</tbody>
</table>

No feasible solution exists $\lambda > 12$

It will be noted that both procedures give the same answers.

**Sensitivity Analysis**

Sensitivity analysis of constraint constants in a linear programming problem, measures the magnitude of change in the value of the constraint constant while maintaining the feasibility of optimum solution.

Sensitivity analysis is, in fact, a special case of parametric programming. When in a parametric model, a single constraint constant is replaced by $\lambda$, the problem can be analyzed for sensitivity.

The derivation for sensitivity analysis with one single parameter in the constraints, follows the same procedure as that for parametric programming. Thus for a minimizing function

$$\max \left(- \frac{\alpha_i}{a_{i1}}\right) \leq \lambda \leq \min \left(\frac{\alpha_i}{a_{i1}}\right)$$

$$a_{i1} > 0 \quad \alpha_i > 0$$

or

$$-\infty \text{ if all } a_{i1} \leq 0 \quad +\infty \text{ if all } a_{i1} \geq 0$$

[3.11]
where $a_{11}$ is the element in the $i$th row and $j$th column of inverse matrix $A^{-1}$.

Gass (6) found the sensitivity range of the constraints as

$$\max \left( -\frac{a_{10}}{a_{11}} \right) \leq \Delta b \leq \min \left( \frac{a_{10}}{|a_{11}|} \right)$$

where $a_{10}$ is the value of the constraint constant in an optimum solution.

The terminology used in [3.11] and [3.12] can be related to each other to satisfy the unifying concept.

$\Delta b_1$ is defined as the change in the value of $1$th constraint in the constant column. In [3.11], this change is called $\lambda$.

In an optimum solution, before any change is made in the constraint constant,

$$x^0 = A^{-1} b \geq 0 \quad [3.13]$$

where

$x^0$ is a column vector $(a_{10}, a_{20}, \ldots, a_{m0})$

for all $a_{10} \geq 0$. 

If a change \( \Delta b_1 \) is made in the value of 1th constant, then [3.13] changes to

\[
\bar{x}_0 = A^{-1} \bar{b} = (a_{i0} + \Delta b_1 a_{i1}) \geq 0
\]

[3.14]

for all \( i \) in the basis and where \( a_{i1} \) is the element in the 1th row and 1th column of \( A^{-1} \).

From [3.14],

\[
\Delta b_1 = \frac{a_{i0}}{-a_{i1}}
\]

[3.15]

but \( a_{i0} \) is \( \alpha \) term and \( a_{i1} \) is \( \beta \) term, therefore

\[
\lambda = \Delta b_1 = -\frac{\alpha_1}{\beta_1}
\]

[3.16]

In [3.15] and finally in [3.16], \( a_{i0} \) was positive to start with, but \( a_{i1} \) can either be positive or negative. The variation of \( a_{i1} \) (\( \beta \)) thus determines either limit.

Therefore relations [3.11] and [3.12] are similar and have been related in terminology.
CHAPTER IV

PARAMETRIC PROGRAMMING AND SENSITIVITY ANALYSIS

WITH TWO PARAMETERS IN OBJECTIVE FUNCTION

The case of one parameter in the objective function of a parametric programming has been studied and a technique of generating additional solutions developed. The solution of a parametric problem tends to get more complex with the increase in the number of parameters in the objective function or constraint constants. A parametric model with two parameters in the objective function is to be considered.

A two parameter objective function problem can either be solved as a two parameter case or can be reduced to a series of single parameter problems by fixing one parameter at a series of constant values. Both these cases will be discussed separately.

A two parameter objective function problem is stated as

Minimize \( \sum_{j=1}^{n} \left( c_j + \lambda c_j + \mu c'' \right) x_j \)

Subject to \( \sum_{j=1}^{n} a_{ij} \cdot x_j = b_i \) \( (i = 1, 2, \ldots, m) \)

\( x_j \geq 0 \) \( (j = 1, 2, \ldots, n) \)

\[ [4.1] \]
where

\( c_j \) is the coefficient of the \( j \)th variable in the objective function.

\( \lambda c_j \) is the first parametric change in the value of \( c_j \).

\( \mu c_j \) is the second parametric change in the value of \( c_j \).

In [4.1] the objective function is expressed as a linear function of \( \lambda \) and \( \mu \). In order to find a minimum feasible solution, it is necessary to determine intervals of values of \( \lambda \) and \( \mu \). As usual, for a minimizing objective function, the necessary precondition is

\[ (z_j - c_j) \leq 0 \quad \text{for all } j \]  \[4.2\]

Further, the presence of \( \lambda \) and \( \mu \) in the objective function shall result in \( \lambda \) and \( \mu \) appearing in the \((z_j - c_j)\) row. The \((z_j - c_j)\) row can be expressed as

\[ \alpha_j + \lambda \beta_j + \mu \gamma_j \]

from [4.2]

\[ \alpha_j + \lambda \beta_j + \mu \gamma_j \leq 0 \quad \text{for all } j \]  \[4.3\]

Each iteration will give a set of linear functions in \( \lambda \) and \( \mu \). These can be plotted in the \( \lambda, \mu \) plane, and a characteristic region can be determined. Since the number of basic feasible solutions is finite, the number of characteristic regions is finite also.
If CR₁ is the characteristic region corresponding to the first minimum feasible solution, to find CR₂, it is necessary to enter a new variable in the solution. The new minimum feasible solution so obtained shall be minimum for at least those points of the corresponding inequality which bound CR₁, that is, there shall be a common boundary between CR₁ and CR₂. Any points on the common boundary shall satisfy both solutions.

The solution is complete when the procedure has systematically exhausted the characteristic regions in λ,μ plane.

It can be noted that the same sort of derivation could be made for the two-dimensional problem as the one-dimensional problem in Chapter II. If values of λ and/or μ appear in more than two columns in the \((z_j - c_j)\) row, only the most restrictive pair of constraints can be used.

It is adequate again to proceed directly to an example.

Minimize \[ z = -(2 + \lambda)x_1 + (\mu + \lambda)x_2 \]

Subject to \[-x_1 + 2x_2 \leq 4 \]
\[x_1 + x_2 \leq 5 \]
\[2x_1 - x_2 \leq 8 \]
\[x_j \geq 0 \]

Following the normal procedure, the initial tableau can be developed.
Tableau 1

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>Basis</th>
<th>( P_0 )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
<th>( P_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective Column</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( m + 1 )</td>
<td>( P_3 )</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( m + 2 )</td>
<td>( P_4 )</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( m + 3 )</td>
<td>( P_5 )</td>
<td>8</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

In this case, the \( P_1 \) and \( P_2 \) vectors have values of \( \lambda \) and \( \mu \).

Therefore the boundaries are expressed by the lines in the \((z_j - c_j)\) rows, that is,

\[
(2 + \lambda) = 0 \\
(- \lambda - \mu) = 0
\]  

The area (in the \( \lambda, \mu \) space) for the initial solution is shown in Figure 4-1.

Selecting the \( P_2 \) vector as the pivot, the second tableau is,
Figure 4-1. Graph showing first characteristic region in $\lambda, \mu$ plane.

$$\begin{align*}
(2 + 2\lambda)^{1/2} &= 0 \\
CR_1 &= 0 \\
x_1 &= 0 \\
x_2 &= 0
\end{align*}$$
Tableau 2

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>Objective Basis</th>
<th>P₀</th>
<th>P₁</th>
<th>P₂</th>
<th>P₃</th>
<th>P₄</th>
<th>P₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +λ)</td>
<td>P₂</td>
<td>2</td>
<td>-1/2</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>P₄</td>
<td>3</td>
<td>3/2</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>P₅</td>
<td>10</td>
<td>3/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(m + 1)</td>
<td></td>
<td>0</td>
<td>+2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(m + 2)</td>
<td></td>
<td>+2λ</td>
<td>+λ/2</td>
<td>0</td>
<td>+μ/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(m + 3)</td>
<td></td>
<td>+2μ</td>
<td>-μ/2</td>
<td>0</td>
<td>+μ/2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

and the boundaries are

\[ 2 + \frac{\lambda}{2} - \frac{\mu}{2} = 0 \]

\[ \frac{\lambda}{2} + \frac{\mu}{2} = 0 \] [4.5]

The new area is shown in Figure 4-b.

In a like manner, when P₁ is the pivot, the third tableau is determined.

Tableau 3

<table>
<thead>
<tr>
<th>Objective Row</th>
<th>Objective Basis</th>
<th>P₀</th>
<th>P₁</th>
<th>P₂</th>
<th>P₃</th>
<th>P₄</th>
<th>P₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +λ)</td>
<td>P₂</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>-(2+λ)</td>
<td>P₁</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1/3</td>
<td>2/3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>P₅</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(m + 1)</td>
<td></td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>-4/3</td>
<td>0</td>
</tr>
<tr>
<td>(m + 2)</td>
<td></td>
<td>+λ</td>
<td>0</td>
<td>0</td>
<td>+2/3λ</td>
<td>-λ/3</td>
<td>0</td>
</tr>
<tr>
<td>(m + 3)</td>
<td></td>
<td>+3μ</td>
<td>0</td>
<td>0</td>
<td>+μ/3</td>
<td>+μ/3</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 4-2. Graph with two characteristic regions in $\lambda, \mu$ plane.
giving a solution of \( x_1 = 2, \ x_2 = 3 \) and characteristic region bounded by

\[
\frac{2}{3} + \frac{2}{3} + \frac{\lambda}{3} = 0 \quad \text{[4.6]}
\]
\[
\frac{4}{3} - \frac{\lambda}{3} + \frac{\mu}{3} = 0
\]

Again Figure 4-3 gives the graphical representation.

The final boundaries are found by pivoting on \( P_4 \) and then \( P_5 \), resulting in Figure 4-4.

From the procedure developed, it would seem that an 'n' parameter problem would require a solution of the boundary hyperplanes in n-dimensions. This has not been further investigated.

The second procedure calls for successive solutions of single dimensional parameter problems.

Effectively, if one of the two parameters in the objective function of a parametric programming is fixed at a constant value, the two parameter case reduces to a series of single parameter problems. The problem is then solved for characteristic intervals for one parameter corresponding to a constant value of the other. These intervals when plotted on a \( \lambda, \mu \) plane, lie on the boundaries of a characteristic region.

When this procedure is repeated for another nearby value of \( \delta \), where \( \lambda = \delta \), another set of values is also defined on the boundaries of the same characteristic region. If these points are located on the \( \lambda, \mu \) plane and if the line containing the related end points are drawn, the lines \( L \) and \( M \) form the boundaries of the characteristic region for the real variables as shown in Figure 4-5.
Figure 4-3. Graph with three characteristic regions in $\lambda, \mu$ plane.
Figure 4-4. Graph showing complete solution of a parametric model with two parameters in objective function.
Figure 4-5. Solution technique for solving a parametric model with two parameters in objective function as a single parameter problem.
If the complete real line for \( \lambda \) and a given \( \mu \) has more than three characteristic intervals, then a point on the boundary lines between more than three characteristic regions has been found. Any given line may not intersect all characteristic regions. At least two characteristic interval lines in \( \lambda \) (i.e. two different fixed values of \( \mu \)) and two characteristic interval lines in \( \mu \) are needed to establish all of the characteristic regions in the \( \lambda, \mu \) plane. If properly selected, they will, in fact, provide two points on the boundary between each characteristic region. This process leads to the determination of all characteristic regions in a systematic manner. Since the number of characteristic regions is finite, the computation shall terminate in a finite number of steps.

Since the range of values of the parameters is from \(-\infty \) to \( +\infty \), most of the characteristic regions are unbounded and convex. This does not eliminate either the possibility of point solutions or bounded regions depending on the number of variables in the problem.

The same problem has been solved again to illustrate this procedure and the values of the parameters have been tabulated. The graph of the \( \lambda, \mu \) plane thus obtained by adopting two different procedures is the same and is shown in Figure 4-4.
TABLE 4-1

OUTPUT TABLE FOR A PARAMETRIC MODEL WITH TWO PARAMETERS IN OBJECTIVE FUNCTION WHEN SOLVED AS A SINGLE PARAMETER CASE

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values of Real Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>10</td>
<td>$-\infty \leq \mu \leq -10$</td>
</tr>
<tr>
<td></td>
<td>$-10 \leq \mu \leq +14$</td>
</tr>
<tr>
<td></td>
<td>$+14 \leq \mu \leq +\infty$</td>
</tr>
<tr>
<td>9</td>
<td>$-\infty \leq \mu \leq -9$</td>
</tr>
<tr>
<td></td>
<td>$-9 \leq \mu \leq +13$</td>
</tr>
<tr>
<td></td>
<td>$+13 \leq \mu \leq +\infty$</td>
</tr>
<tr>
<td>-2</td>
<td>$-\infty \leq \mu \leq 2$</td>
</tr>
<tr>
<td></td>
<td>$2 \leq \mu \leq 2$</td>
</tr>
<tr>
<td></td>
<td>$2 \leq \mu \leq +\infty$</td>
</tr>
<tr>
<td></td>
<td>$2 \leq \mu \leq 4.33$</td>
</tr>
<tr>
<td></td>
<td>$-\infty \leq \mu \leq 2$</td>
</tr>
<tr>
<td>-6</td>
<td>$-\infty \leq \mu \leq 4$</td>
</tr>
<tr>
<td></td>
<td>$4 \leq \mu \leq 10$</td>
</tr>
<tr>
<td></td>
<td>$10 \leq \mu \leq +\infty$</td>
</tr>
<tr>
<td>-7</td>
<td>$-\infty \leq \mu \leq 4.25$</td>
</tr>
<tr>
<td></td>
<td>$4.25 \leq \mu \leq 12$</td>
</tr>
<tr>
<td></td>
<td>$12 \leq \mu \leq +\infty$</td>
</tr>
</tbody>
</table>
A parametric model is obtained when one or more known coefficients in a linear programming problem are replaced by parameters.

When the single parameter is in the objective function in programming problems, the range of values of the parameter $\lambda$ can be determined for every basic feasible solution. Every basic feasible solution might be expected to be optimum for at least one value of $\lambda$, but there always is a point that is antithesis of the objective function. Once an optimum solution is obtained for a given value of $\lambda$, the solution of the ranges of $\lambda$ can be systematically obtained.

Sensitivity analysis has been demonstrated as a special case of parametric programming in which the parameter $\lambda$ replaces only one cost coefficient in the objective function.

A parametric model with a parameter in the constraint constants, can be solved by means of specific comparisons from the $(z_j - c_j)$ row or can be converted, via the dual form, to a one parameter objective function problem and can be solved as such for all values of the parameter.

Certain other properties of $\lambda$, the parameter, are given below.
1. Each range of values of $\lambda$ establishes an optimum solution.
2. The values of $\lambda$ corresponding to a solution may be either a point or an interval. The interval may be closed or unbounded in either direction.
3. The number of intervals is finite.
4. Two intervals must always meet at a point.
5. The collection of intervals forms a connected set.

A parametric problem gets increasingly complex with the increase in the number of parameters. A procedure has been developed for solving a problem with two parameters in the objective function. It is suggested that it can be generalized to n parameters.

Specific Conclusions
1. For a given objective function, there always is a point that is the antithesis of the objective function.
2. Sensitivity analysis has been demonstrated as a special case of parametric programming where the parameter replaces only one coefficient in the objective function.
3. A parametric problem with a parameter in the constraints when written in dual form, is a one parameter objective function problem and can be solved as such.
4. It is possible to solve a two parameter problem by using the single parameter procedure and the range of total solution can be graphically demonstrated on the plane of the parameters.
5. In the two dimensional parametric programming the parameters are linearly related to each other.
6. Since the number of basic feasible solutions is finite, the number of characteristic regions is finite too. All characteristic regions represent convex sets and two characteristic regions have at most a boundary in common.
7. Since the range of interest in the values of \( \lambda \) and \( \mu \) is \(-\infty\) to \(+\infty\), all the characteristic regions may be unbounded in a two variable problem.

Recommendations

1. Solution of a parametric model with two parameters in the constraints should be possible if the model is written in dual form and solved as a two parameter objective function model.

2. A generalized procedure for an n-dimensional parametric problem should be developed. The two dimensional procedure can be extended, at least theoretically, to include procedure for n-dimensional model, but has not been proven.
BIBLIOGRAPHY


