Forced Vibrations of Hollow Cylindrical Structures

Douglas Jose Nunez Briceno

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FORCED VIBRATIONS OF HOLLOW CYLINDRICAL STRUCTURES

BY

DOUGLAS JOSE NÚÑEZ BRICÉÑO

A thesis submitted in partial fulfillment of the requirements for the degree
Master of Science
Major in Mechanical Engineering
South Dakota State University
1988
To the memory of my beloved father

Benigno Antonio Núñez Duran
FORCED VIBRATIONS OF HOLLOW CYLINDRICAL STRUCTURES

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable for meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Hamid R. Hamidzadeh
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Head, Department of Mechanical Engineering
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</tr>
<tr>
<td>b</td>
<td>Outer radius of the cylinder</td>
</tr>
<tr>
<td>E</td>
<td>Modulus of Elasticity</td>
</tr>
<tr>
<td>G</td>
<td>Shear Modulus of Elasticity</td>
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<tr>
<td>H</td>
<td>Thickness of the cylinder</td>
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<td>$I_n$</td>
<td>Modified Bessel Function of the first kind</td>
</tr>
<tr>
<td>$J_n$</td>
<td>Bessel Function of the first kind</td>
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<tr>
<td>$K_n$</td>
<td>Modified Bessel Function of the second kind</td>
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<tr>
<td>L</td>
<td>Axial half wavelength</td>
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<td>n</td>
<td>Number of circumferential waves</td>
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<td>P</td>
<td>Exciting frequency</td>
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<td>R</td>
<td>Mean radius of the cylinder</td>
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<tr>
<td>r, z, $\theta$</td>
<td>Cylindrical coordinates</td>
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<td>$\varepsilon_{ij}$</td>
<td>Components of strain</td>
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<td>$\gamma$</td>
<td>Wave number in axial direction</td>
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<td>$\omega_s$</td>
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LIST OF SYMBOLS
\[ \Omega = \frac{P}{\omega_s} = \text{Normalized frequency factor} \]

\[ \rho = \text{Density} \]

\[ \lambda = \text{Lame's constant} \]

\[ \nu = \text{Poisson's ratio} \]

\[ \eta = \text{Structural damping factor} \]

I am most grateful for his invaluable guidance, advice, encouragement and dedication throughout evolving my work.

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CHAPTER ONE
INTRODUCTION

1.1 Historical Study of the Importance of Vibration

The first individual to show any interest in the study of vibrations was the Italian Galileo Galilei (1564-1642), who by using geometrical means showed the dependence of natural frequency of a simple pendulum on the pendulum's length. Galileo also initiated the study of vibrations of rings and plates, but he could not establish an analytical treatment which could explain his experimental observations. A whole series of pages and chapters could be and have been devoted to the historical development of the field of vibration. For the reader's reference, it is worthwhile to mention a few books which have chapters dealing with this topic: Timoshenko's History of Strength of Materials, (1953), or J.T. Cannon's Evolution of Dynamics: Vibration Theory from 1687 to 1742, (1981) or Love's Treatise on the Mathematical Theory of Elasticity, (1944). This introductory chapter by no means attempts to provide a detailed account of the historical development in vibrations. However, in order to better illustrate the important role that the study of vibrations has played in the past and will play in the future of the engineering field, the author feels that a brief outline of the major developments on vibration research is imperative at this point.
After Galileo’s pioneer studies, Joseph Sauveur (1653-1716) studied the vibrations of strings. It was Sauveur who first calculated the frequencies of a vibrating string as a function of the string’s sag at its center. It was also Sauveur who first used the term “nodes” to indicate zero displacement points on a string vibrating at its natural frequency. He also used the terms “fundamental” for the lowest natural frequency and “harmonics” for the other frequencies.

As other areas of science were being developed, the study of vibration became more precise. Three main scientific discoveries which contributed to the enhancement of the study of vibrations were: (1) the establishment of the basic law of elasticity by Robert Hooke in 1676; (2) the enunciation of Newton’s law relating force, mass and acceleration in 1687; and (3) the establishment of differential calculus by Leibniz (1646-1716).

Among the many notable names contributing to the study of vibrations, it is imperative to include Taylor, Bernoulli, D’Alembert, Euler, Lagrange, Kirchoff, Fourier, Lamb, Coulomb and Rayleigh. In 1877, Rayleigh published his results on the theory of sound. He also proposed the use of the principle of conservation of energy to find the fundamental frequency of vibrating conservative systems.

Among the pioneers of vibration research in the present century, the following stand out: Stodola (vibrating turbine blades), Timoshenko (vibrations of beams and plates), and Mindlin (vibrations of beams and plates), Minorsky and Stoker (non-linear vibration).

Whereas the pioneers of vibration investigations sought the
understanding of the natural phenomena and the derivation of mathematical models to better describe the vibration of physical systems, the researchers in this century have strived to apply the knowledge gained in the past to make new discoveries, so that vibration analysis can be applied to practical engineering problems. Hence, today's engineers have become aware of the fact that many engineering systems are subjected to vibrations either because of external excitations (forced vibrations) or because of the system's ability to store energy as a result of its elastic properties (free vibrations). Being able to design so vibrational effects are minimized has become a fundamental asset in today's engineering applications. As a result, the vibration field has been able to grow and become more specialized. Vibration analysis is now capable of playing a very important role in a wide range of engineering applications such as the design of machines, engines, foundations, control systems and structures.

For instance, an engineer can use vibration knowledge to enhance the design of vital structures (such as bridges, nuclear reactors, and buildings), particularly when earth movement is a concern. Since World War II, the results obtained from the vibration studies of cylindrical structures have successfully been used, but are not limited to improving the design of submarines, pipelines, missile boosters and compressor shells. A mechanical engineer can apply vibration concepts to design efficient damping treatments used to minimize the vibration effects of many machinery components. The unbalances in rotating machinery can create oscillations that bring about the failure of machine components due to fatigue. In other
cases, these unbalances may create ground waves or excessive noise which can produce discomfort to humans. In either case, the problem can be considered by using vibration analysis. The few examples mentioned above are indications of the important role the theory of vibrations has played and will continue to play in the engineering discipline.

1.2 Vibrations of Solid Circular Rods

In the previous section, the significance of the general field of vibrations was considered. Since this investigation is restricted to the study of vibrations of circular structures, the remainder of this introductory chapter reviews and categorizes past investigations regarding the vibrations of circular structures.

Since the vibrations of an elastic solid includes the transmission of elastic energy by two types of waves (dilatational and distortional), the study of vibrations requires the analysis of wave propagation. Each type of wave propagates at a speed which is the ratio of the inertia (density) and deformability (elastic constants) of the solid. However, when an elastic body is subjected to vibrations, both types of waves are generated, and as a consequence, the rate of wave propagation is no longer constant. In fact, it depends on the wavelength of the excitation. Solving the resulting wave propagation problem in a bounded elastic medium using the three-dimensional theory of elasticity becomes a difficult task. Many methods of solution to this problem have been presented, a summary of
which follows. Throughout the years, studies of vibrating circular structures have been divided into three general categories: vibrations of elastic solid cylinders, vibrations of isotropic elastic hollow cylinders, and vibrations of hollow cylinders made of viscoelastic materials. This research is mainly concerned with the study of vibrations of elastic hollow cylinders.

The first mathematical attempt to study the vibrations of solid cylinders was conducted by Pochhammer (1876), who employed the general theory of elasticity in his analysis. Pochhammer was followed by Chree (1886). Both Pochhammer and Chree considered an infinitely long circular rod which was traction-free on its circumferential boundary, but their solution did not allow traction-free ends. After Pochhammer and Chree many analytical techniques were presented. However, these techniques assumed the solid cylinder was either a thin disk or a slender rod, and, as a result, were inadequate to predict the natural frequencies of solid cylinders when the height-to-radius ratio approached unity.

McNiven and Perry (1962) considered the coupling of longitudinal axial shear, and radial modes of propagation in an infinite length rod. The Rayleigh-Ritz method was used by Rumerman and Raynor (1971) to study the radial and axial modes of an infinite cylinder. The following year, J.R. Hutchinson (1972) proposed a series solution of the general three-dimensional theory of elasticity equations to study the axisymmetric vibrations of a free finite length rod.

G.W. McMahon (1964) published the results of his
experimental studies of solid aluminum and steel cylinders with free boundaries and with height-to-radius ratios ranging from 0.2 to 3.3. Using a shaker, he excited six modes of circumferential order zero, seven modes of order one, four modes of order two, and three modes of order four. The nodes and antinodes formed during vibration were determined by means of sand patterns and vibration probes.

Rasband (1975) extended Hutchinson's work to study the nonaxisymmetric vibrations of a free finite length solid cylinder. However, Rasband did not present any numerical results in his paper. Hutchinson (1980) presented an extension of his earlier work to include both the axisymmetric and nonaxisymmetric vibrations of a free solid rod. Hutchinson's results were in agreement with the experimental results obtained by McMahon (1964).

1.3 Vibrations of Hollow Cylinders

Throughout the years, numerous approaches and techniques have been used to study the vibrations of hollow circular structures. Rayleigh (1894) developed an expression for the natural frequencies of thin cylinders with free ends, considering axial vibration. The flexural vibration of cylinders was considered by Love (1927). Flugge (1934) extended Love's work to obtain a frequency equation for a cylinder with free ends. Arnold and Warburton (1949) used Timoshenko's strain relations to derive frequency equations to verify their experimental results in thin cylinders with freely supported ends. One of the conclusions of their investigation was that for
short cylinders with very thin walls the natural frequencies decrease as the number of circumferential nodes increases. Arnold and Warburton (1949) concluded that this decrement on natural frequencies was attributed to the strain energy dependence on bending and stretching. Arnold and Warburton (1953) extended their studies on flexural vibrations of thin cylinders to include other end conditions, and a range of cylinder thickness, by defining a wavelength factor to account for different end conditions.

Forsberg (1964) used Flugge's three equations of motion to derive a method for determining natural frequencies for uniform thin cylindrical shells regardless of end conditions. Forsberg's method required the initial assumption of a natural frequency. Once this frequency was assumed, the length of the cylindrical shell for given end conditions was derived. Hence, this method was the reverse of all the previous methods in which the natural frequency for a given cylinder's length was sought. The weakness of Forsberg's method consisted in the much greater computation required on comparison with all the previous techniques. Warburton (1965) presented an outline of the general Forsberg's method as applied to thin cylindrical shells for clamped and free ends. Warburton (1965) compared these results with previous approximated theories, and he noticed that his results agreed for long axial wavelengths and large numbers of circumferential waves. He concluded that for long axial wavelengths and large numbers of circumferential waves the effect of end conditions was negligible, and it was possible to treat the ends as simply supported for frequency calculations.
As more applications in the engineering field were found for circular structures, more time was devoted to their study and understanding. As described in the previous paragraphs, many studies used the theory of shells and plates to describe the vibrations of circular structures. From the late 1800's to the present numerous investigations using this theory have been presented. An account of which can be found in the introductory chapter in *Thin-Shell Structures: Theory, Experiment and Design*, edited by Y.C. Fung and E.E. Sechler in 1972.

Although many cylindrical structures can be analyzed using the theory of thin shells, thicker cylinders have to be studied using the general three dimensional theory of elasticity which applies equally to thick or thin shells. It appears that the first individuals to study the vibrations of an infinitely long traction-free hollow cylinder using the linear three-dimensional theory of elasticity were independently Greenspon (1957) and Gazis (1959).

Gazis (1959), in particular, considered the transmission of elastic energy by means of two types of waves (dilatational and distortional) in an infinite isotropic hollow cylinder. By using the three dimensional theory of elasticity, he was able to solve the associated eigenvalue problem for stress-free cylindrical surfaces. He presented tables of natural frequencies for different mean radius-to-thickness ratios (0.01 to 2.0) and for different numbers of circumferential waves (0 to 4) and different thickness to length ratios (0 to 1.0).

After Gazis' investigation, the axisymmetric vibrations of hollow cylinders was considered by McNiven, Shan and Sackman
(1966) using a "three mode theory". The name "three mode theory" originated from the fact that only motions associated with the three lowest modes were considered. This assumption can be justified by realizing that the second mode (radial mode) and the third mode (axial shear mode) have a high influence on the fundamental mode (flexural bending) and on each other. However, the fourth and all higher modes have significantly less influence on the first three modes. Subsequently, the motions associated with the fourth and all higher modes were not considered in the analysis presented by McNiven, Shah and Sackman (1966). Their results showed good agreement with results obtained from the exact three-dimensional theory.

Gladwell and Vijay (1975) studied the three-dimensional vibrations of a finite length circular cylinder with traction-free surfaces for the first time, using a finite element approach. Furthermore, a two-dimensional analysis was carried out by Hamidzadeh (1981) to study the lobar vibrations of thick elastic and viscoelastic infinitely long cylinders, as well as circular thick rings, subjected to uniform harmonic excitations along the cylinder's axis. The resonant frequencies obtained by Hamidzadeh (1981) were in good agreement with the natural frequencies obtained by Gazis (1959) using a free vibration approach. As expected, Hamidzadeh's results differed from the estimated resonant frequencies based on the Timoshenko theory. This is due to the fact that the Timoshenko theory is only valid for thin rings, while Hamidzadeh's technique puts no restrictions on the ring thickness.
P.A. Svardh (1984) published the results of his work on axisymmetric wave propagation in a semi-infinite, hollow, elastic circular cylinder with traction free lateral surfaces initially at rest and subjected to transient end loadings. Two cases were considered: a prescribed axial velocity at the end which was free from shear stresses, and an axial pressure applied to a radially clamped end. A Fourier-Laplace double transform technique was used to obtain asymptotic solutions valid at large distances from the ends for two types of time dependence for the end conditions: step function and finite rise time function. J.R. Hutchinson and S.A. El-Azhari (1986) extended Hutchinson's work in solid cylinders to include free hollow cylinders with finite length. They used a series solution of the general three dimensional theory of elasticity to find the natural frequencies. Their technique showed agreement with the results given by Gladwell and Vijay (1975), and it also agreed with the Timoshenko theory for beams of thin walled cross-section.

1.4 Damping Considerations

In the preceding sections, a literature review of past investigations regarding the determination of natural frequencies for circular structures (solid and hollow) has been presented. Knowing the natural frequencies of a vibrating system is of vital importance to the vibration analyst because it provides the needed information to determine the resonance condition of the system, which is the frequency producing the highest amplitude of vibration in the system.
However, another important consideration in vibration analysis is vibration attenuation or damping treatment.

The main objective in damping treatment is the incorporation of energy dissipating devices in the system. The ideal solution is the use of materials with high hysteretic (structural) damping. In general, materials with high structural damping coefficients have low structural strength, and as a result they are not suitable for many engineering applications where structural strength is required.

The use of viscoelastic materials instead of pure elastic materials results in lower vibration amplitudes. An alternative approach is the use of a combination of two or more materials in alternate layers. Usually two different materials are used; one material provides the desired strength of the member, and the second provides the desired structural damping. The latter can be called a viscoelastic material, and it dissipates energy in the form of heat when subjected to alternating stresses.

The viscoelastic damping treatment described in the previous paragraph has been used primarily in two types of arrangements, the free viscoelastic layer and the constrained or sandwich layer. In the free viscoelastic layer arrangement, a surface of the viscoelastic material is attached to a high strength material, and the other surface is free. In this case, energy dissipation occurs when the viscoelastic material is extended or compressed. As the name implies, constrained layer damping is accomplished by sandwiching a viscoelastic material between two high strength materials. In this case, the energy dissipation is mainly due to the shear deformation of the
viscoelastic layer. H.R. Hamidzadeh (1981, 1982) showed that the constrained sandwich layer damping treatment is more effective than the free viscoelastic layer damping treatment.

Not surprisingly, Timoshenko's curved beam theory has been utilized in damping treatments of cylindrical structures. Furthermore, Timoshenko's ring theory provides breathing (lobar) nodes frequencies for thin-layered cylindrical structures.

Any further discussion of viscoelastic damping is beyond the scope of this investigation. However, for an extensive literature review in damping treatment see publications by Agbasiere and Grootenhuis (1968), Grootenhuis (1970), Nakra and Grootenhuis (1972, 1974), Ioannides and Grootenhuis (1979) and Lu and Everstine (1980).

1.5 Scope of the Present Investigation

As it can be seen in the literature survey given in the previous paragraphs, many research studies have been published describing the vibration of circular structures by using the shell theory approach. Although it can be employed in the analysis of many engineering systems, the shell theory has proven to be inadequate in the analysis of systems which can not be modeled as thin shells. It has been shown by Gazis (1959), Hutchinson (1972, 1986), and Hamidzadeh (1981, 1982) that the thin shell theory can not provide the dynamic response adequately. This is due to the fact that the thin-shell theory does not have the capability of considering
thickness modes. A more realistic technique is the use of the general three-dimensional theory of elasticity which puts no restrictions on the cylinder's thickness or length.

In the past, elastodynamic investigations in circular structures have been limited to the study of free vibration problems. In those investigations the general approach was to use the theory of elasticity to set up a homogeneous system of equations for stresses and displacements. In order for a system of homogeneous equations to have non-trivial solutions, it is required that the determinant of its coefficient matrix be equal to zero. A sweeping procedure was employed to find the frequencies at which the coefficient determinant vanished. These frequencies represented the natural frequencies of the system.

Since most of the engineering systems are subjected to forced vibrations rather than free vibrations, this investigation is solely concerned with the solution of a forced vibration problem. The general three-dimensional theory of elasticity is used to obtain mathematical relations for stress and displacements.

The governing equations of motion in terms of volumetric strain and three elastic rotations for a homogeneous, isotropic elastic medium are derived in chapter two. These governing equations are derived from the equilibrium and compatibility relations based on the theory of elasticity. Since the present investigation is concerned with the vibrations of elastic cylinders, the aforementioned equations are derived using cylindrical coordinates (r, θ and z).

After satisfying the boundary conditions of a hollow
Furthermore, a comparison is made between the natural frequency factors obtained by Gazis (1959) and the resonant frequency factors computed in this investigation. As shown in Tables 4.1 to 4.25, the two approaches agree very closely for small values of H/L. However, they give very different values for the fundamental frequency when H/L is greater than 0.50.
cylinder subjected to an harmonic external excitation, a system of non-homogeneous linear equations for stresses and displacements is obtained. This system of equations can be solved for the stresses and displacements for a wide range of exciting frequencies. By observing which exciting frequencies produce the highest amplitudes for stresses and displacements the resonant frequencies can be established. Since resonance occurs when the natural frequency of the system is equal to the exciting frequency, the natural frequencies of the system can be determined.

The choice of a harmonic form of excitation can be justified by the fact that the solution of this type of problem allows the use of a sweeping procedure for the computation of the resonant frequencies. Furthermore, in practical applications, a general periodic disturbance or even a single pulse disturbance can be modeled by a superposition of a series of harmonic excitations (Fourier Analysis).

In this investigation, a wide range of thickness modes is studied; design charts for computing natural frequencies for a wide range of thickness ratio (H/R) as well as for different ratios of thickness to length of cylinder (H/L) are provided for axisymmetric vibration, rigid body motion and the first three lobar modes. Moreover, the variation of the first resonant frequency with respect to Poisson's ratio was analyzed at five different numbers of circumferential waves for a typical thickness ratio and several length's ratio. In addition, the effect of structural damping in the displacements amplitude was studied at the first resonant frequency for a typical cylinder's geometry.
The main purpose of this chapter is to derive the governing equations for stresses, strains, elastic rotations and displacements for a homogeneous, isotropic elastic medium. These relations are derived from the equilibrium and compatibility equations based on the Theory of Elasticity. The mathematical relations derived in this chapter will be used in the coming chapters to present a mathematical model to the problem studied in this investigation.

2.1 State of Stresses for an Infinitesimal Element

The state of stresses for an infinitesimal element in cylindrical coordinates is shown in Figure 2.1. In Figure 2.1, \( \sigma_{ii} \) represents a direct stress along coordinate \( i \), and \( \tau_{ij} \) indicates a shear stress along coordinate \( j \) and perpendicular to coordinate \( i \). The symbols \( \delta r, \delta z \) and \( \delta \theta \) represent the lengths of the infinitesimal element in the directions of \( r, z \) and \( \theta \). The displacements in the \( r, \theta, z \) direction are represented by \( u, v \) and \( w \) respectively.
The prime notation for $\sigma$ and $\tau$ are given by the following equations:

\[
\sigma_{rr}' = \sigma_{rr} + \frac{1}{r} \left( \frac{\partial \sigma_{\theta\theta}}{\partial r} \right) r
\]

\[
\sigma_{\theta\theta}' = \sigma_{\theta\theta} + \frac{1}{r} \left( \frac{\partial \sigma_{rr}}{\partial \theta} \right) r
\]

\[
\sigma_{zz}' = \sigma_{zz}
\]

\[
\tau_{rz}' = \tau_{rz} + \frac{1}{r} \left( \frac{\partial \tau_{rz}}{\partial r} \right) r
\]

\[
\tau_{rr}' = \tau_{rr} + \frac{1}{r} \left( \frac{\partial \tau_{\theta\theta}}{\partial r} \right) r
\]

\[
\tau_{\theta\theta}' = \tau_{\theta\theta} + \frac{1}{r} \left( \frac{\partial \tau_{rr}}{\partial \theta} \right) r
\]

As shown in Figure 2.1, six stress components are required to specify the state of stresses of an infinitesimal element. In the absence of body or surface couples the equilibrium of moments for the infinitesimal volume element yields:

\[
t_{rr} = t_{\theta\theta} = t_{zz} = 0
\]

\[
t_{rz} = t_{rz}' = 0
\]

Hence, under these conditions only six stress components are needed.

**Figure 2.1** State of stresses for an infinitesimal element in cylindrical coordinates
The prime notation for $\sigma$ and $\tau$ are given by the following equations:

\[
\sigma'_{rr} = \sigma_{rr} + \left(\frac{\partial \sigma_{rr}}{\partial r}\right) r \delta r \\
\sigma'_{\theta\theta} = \sigma_{\theta\theta} + \left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right) \theta \delta \theta \\
\sigma'_{zz} = \sigma_{zz} + \left(\frac{\partial \sigma_{zz}}{\partial z}\right) z \delta z \\
\tau'_{zr} = \tau_{zr} + \left(\frac{\partial \tau_{zr}}{\partial z}\right) (r \delta r) \\
\tau'_{\theta r} = \tau_{\theta r} + \left(\frac{\partial \tau_{\theta r}}{\partial \theta}\right) (r \delta r) \\
\tau'_{\theta z} = \tau_{\theta z} + \left(\frac{\partial \tau_{\theta z}}{\partial \theta}\right) (z \delta z) \\
\tau'_{r\theta} = \tau_{r\theta} + \left(\frac{\partial \tau_{r\theta}}{\partial r}\right) (r \delta r) \\
\tau'_{rz} = \tau_{rz} + \left(\frac{\partial \tau_{rz}}{\partial r}\right) (r \delta r)
\]

As shown in Figure 2.1, nine stress components are required to specify the state of stresses of an infinitesimal element. In the absence of body or surface couples the equilibrium of moments for the infinitesimal volume element yields:

\[
\tau_{r\theta} = \tau_{\theta r} \\
\tau_{\theta z} = \tau_{z\theta} \\
\tau_{rz} = \tau_{rz}
\]

Hence, under these conditions only six stress components are needed.
to specify the state of stresses of an infinitesimal element.

2.2 Equilibrium Equations in Terms of Stresses

It is understood that the stress components of the element shown in figure 2.1 varies from one point to another in the element. These changes in stresses are given by equations 2.1. Furthermore, according to Newton's laws of motion one equilibrium equation can be written in each direction of the cylindrical coordinates. In addition to the stresses shown in figure 2.1, body forces acting throughout the element need to be considered. Hence, $F_r$, $F_\theta$, and $F_z$ are introduced to be the body forces per unit volume in the $r$, $\theta$, and $z$ directions. Referring to figure 2.1, summation of the forces acting in the $r$, $\theta$ and $z$ directions results in:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\partial \sigma_{rr}}{\partial r} - \sigma_{\theta\theta} + F_r = 0 \tag{2.2a}
\]

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0 \tag{2.2b}
\]

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{zz}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0 \tag{2.2c}
\]

A detailed derivation of the above three equations can be found in classical publications on the Theory of Elasticity such as Southwell (1944), Ford (1963) and Love (1944).
2.3 Stress-Strain Relations

For a homogeneous and isotropic material - a material having the same properties in all directions - Hooke's law can be stated as:

\[ \sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz}) = E\varepsilon_{rr} \quad (2.3a) \]
\[ \sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz}) = E\varepsilon_{\theta\theta} \quad (2.3b) \]
\[ \sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr}) = E\varepsilon_{zz} \quad (2.3c) \]
\[ \tau_{r\theta} = G\gamma_{r\theta} \quad (2.4a) \]
\[ \tau_{rz} = G\gamma_{rz} \quad (2.4b) \]
\[ \tau_{\theta z} = G\gamma_{\theta z} \quad (2.4c) \]

where:

- $E$ = Young's modulus of elasticity
- $G = \frac{E}{2(1+\nu)}$ = shear modulus of elasticity \hfill (2.5a)
- $\nu$ = Poisson's ratio
- $\gamma_{ij}$ = shear strain along $j$ and perpendicular to $i$
- $\varepsilon_{rr}$, $\varepsilon_{\theta\theta}$, $\varepsilon_{zz}$ = strains in the directions of $r$, $\theta$ and $z$

Lame's elastic constant and volumetric strain are introduced to be:

\[ \lambda = \frac{\nu E}{(1-2\nu)(1+\nu)} \quad \text{(Lame's constant)} \quad (2.5b) \]
\[ \varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} \]  
(volumetric strain)  

(2.5c)

Adding equations 2.3a, 2.3b and 2.3c gives:

\[ E\varepsilon = (1 - 2\nu)(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) \]  

(2.5c)

\[ \frac{E\varepsilon}{1 - 2\nu} = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} \]  

(2.5c)

From equation 2.3a

\[ \sigma_{\theta\theta} + \sigma_{zz} = \frac{\sigma_{rr} - E\varepsilon_{rr}}{\nu} \]  

(2.5d)

\[ \frac{E\varepsilon}{1 - 2\nu} = \sigma_{rr} + \frac{\sigma_{rr}}{\nu} - \frac{E\varepsilon_{rr}}{\nu} \]  

(2.5e)

\[ \frac{\nu E\varepsilon}{1 - 2\nu} = (1 + \nu)\sigma_{rr} - E\varepsilon_{rr} \]  

(2.5f)

Then

\[ \sigma_{rr} = \frac{\nu E\varepsilon}{(1 - 2\nu)(1 + \nu)} + \frac{E\varepsilon_{rr}}{1 + \nu} \]  

(2.5g)

and from equations 2.5a and 2.5b

\[ \sigma_{rr} = \lambda \varepsilon + 2G \varepsilon_{rr} \]  

(2.6a)
Using the same procedure outlined above gives:

\[ \sigma_{\theta\theta} = \lambda \varepsilon + 2G \varepsilon_{\theta\theta} \] (2.6b)

\[ \sigma_{zz} = \lambda \varepsilon + 2G \varepsilon_{zz} \] (2.6c)

Also, the shear stresses in terms of shear strains are given by equations 2.4 as:

\[ \tau_{r\theta} = G \gamma_{r\theta} \] (2.6d)

\[ \tau_{rz} = G \gamma_{rz} \] (2.6e)

\[ \tau_{\theta z} = G \gamma_{\theta z} \] (2.6f)

Equations 2.6 represent the three dimensional Hooke's Law.

2.4 Strains in Terms of Displacements

Since strains generally vary from point to point, its mathematical definition must correspond to an infinitesimal element. During straining every point experiences small displacements in three directions. As shown in figure 2.2, A is a point with coordinates \((r, z, \theta)\) and F is a point having coordinates \((r + \delta r, z + \delta z, \theta + \delta \theta)\), where \(\delta r, \delta z\) and \(\delta \theta\) are infinitesimal changes in \(r, z\) and \(\theta\). When small strains are applied to the body, the volume element is displaced and its shape is distorted as shown in Figure 2.3 below. In Figure 2.3 the new positions of points A, B, C, D, E, F, G, H are represented by the prime notation.
Since the element under consideration is assumed to be infinitesimally small, subjected to small strains, the segments AB, CD, EF, and GH in Figure 2.2 can be taken as straight lines without significant loss in the analysis. Noticing that in general the displacements of F to F' due to the strains $u$ and $v$ through the horizontal plane ACDB and the distorted plane $A'B'C'D'$ be shown as in Figure 2.4 below.

**Figure 2.2** Infinitesimal element in cylindrical coordinates

**Figure 2.3** Distorted infinitesimal element due to straining

**Figure 2.4** Distorted infinitesimal element $A'B'C'D'$ due to straining.
Since the element under consideration is assumed to be infinitesimally small, subjected to small strains, the segments AB, CD, EF and GH in Figure 2.3 can be taken as straight lines without significant loss in the analysis. Noticing that in general the displacements of F to F' are $u + \delta u$, $v + \delta v$ and $w + \delta w$, the horizontal plane ACDB and the distorted plane $A'C'D'B'$ can be shown as in Figure 2.4 below.

Figure 2.4 Distorted horizontal plane $A'C'D'B'$ due to straining of an infinitesimal element

In order to find the radial strain, it is sufficient to consider the strain in the infinitesimal length AC, ignoring the effects of strains in the z direction. Hence,

$$\varepsilon_{rr} = \frac{A_2C_2 - AC}{AC} = \frac{\frac{2u}{\delta r} - \delta r}{\delta r}$$
\[ \varepsilon_{rr} = \frac{\partial u}{\partial r} \tag{2.7a} \]

Similarly the circumferential strain is given by:

\[ \varepsilon_{\theta\theta} = \frac{A'B' - AB}{AB} \]

where:
\[ AB = r\delta\theta \]

and:
\[ A'B' = (r + u)(\delta\theta + \frac{1}{r} \frac{\partial v}{\partial \theta} \delta\theta) \]

The angle of arc A'B' is increased by:

\[ \frac{\partial v}{\partial (r+u)} = \frac{1}{r} \frac{\partial v}{\partial \theta} \delta\theta \]

\[ \varepsilon_{\theta\theta} = \frac{r\delta\theta + (\partial v/\partial \theta)\delta\theta + u\delta\theta + (u/r)(\partial v/\partial \theta)\delta\theta - r\delta\theta}{r\delta\theta} \]

\[ \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{u\partial v}{r^2\partial \theta} \tag{2.7b} \]

If the last term in the above equation is neglected, the circumferential strain can be written as:

\[ \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \tag{2.7b} \]

The strain in the z direction is given as in the cartesian coordinates,
by the expression:

\[ \varepsilon_{zz} = \frac{\partial w}{\partial z} \]  

(2.7c)

From figure 2.4, it can be seen that:

\[ \gamma_{r\theta} = \text{angle } C_3 A'C' + \text{angle } B_1A'B' \]

but

\[ \text{angle } C_3 A'C' = \text{angle } C_1 A'C' - \text{angle } C_1 A'C_2 \]

\[ \text{angle } C_1 A'C_3 = \text{angle } A0A' \]

\[ \left( \frac{\partial v}{\partial r} \right) \]

\[ \text{but } C_1 C' = \frac{\partial v}{\partial r} \text{ and } \text{angle } C_1 A'C' = C_1 C' / \partial r \]

finally:

\[ \text{angle } C_3 A'C' = \frac{\partial v}{\partial r} - \frac{v}{r} \]

Similarly

\[ \gamma_{r\theta} = \text{angle } D_3 A'G' + \text{angle } B_1A'B' \]

\[ \text{angle } B_1A'B' = \left( \frac{B_1B'/r\delta \theta}{r} \right) = \left( \frac{1}{r\delta \theta} \frac{\partial u}{\partial \theta} \delta \theta \right) = \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right) \]

and the shear strain can be written as:

\[ \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \]  

(2.7d)
To find the shear strain $\gamma_{\theta z}$, the vertical planes $ABGH$ and $A'B'G'H'$ shown in Figure 2.5 need to be considered.

![Diagram of distorted vertical plane](image)

**Figure 2.5** Distorted vertical plane $A'B'G'H'$ due to straining of an infinitesimal element

Noticing that the shear strain ($\gamma_{\theta z}$) in this plane is represented by the change from the right angle $BAG$ to the new angle $B'AG'$ results in:

$$\gamma_{\theta z} = \text{angle } G_1A'G' + \text{angle } B_1A'B'$$

$$\gamma_{\theta z} = \frac{1}{\delta z} \frac{\partial v}{\partial z} \delta z + \frac{1}{\delta \theta} \frac{\partial w}{\partial \theta} \delta \theta$$

Furthermore, the non-homogeneous components in the cylindrical coordinates:

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta}$$
Similarly, referring to Figure 2.3, the shear strain $\gamma_{rz}$ is found by examining the planes GACE and G' A'C'E.

$$\gamma_{rz} = \text{angle } G_1 A'G' + C_1 A'C'$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

In summary, the relations between strains and displacements in cylindrical coordinates are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}$$  \hspace{1cm} (2.7a)

$$\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}$$  \hspace{1cm} (2.7b)

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$  \hspace{1cm} (2.7c)

$$\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$  \hspace{1cm} (2.7d)

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta}$$  \hspace{1cm} (2.7e)

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$  \hspace{1cm} (2.7f)

Furthermore, the rotations components in the cylindrical coordinates
are given in Ford (1963) as:

\[
2\omega_r = \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z} \quad (2.8a)
\]

Also, the direct stress in the direction of \(r\) can be obtained from equations 2.6c and 2.7a as:

\[
2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \quad (2.8b)
\]

Combining equations 2.6a, 2.6b, 2.7a and 2.7b with equations 2.7d, 2.7e and 2.7f respectively, the shear stresses in terms of displacements can be presented as:

\[
2\omega_z = \frac{\partial v}{\partial r} + \frac{\partial w}{\partial r} - \frac{1}{r} \frac{\partial u}{\partial \theta} \quad (2.8c)
\]

2.5 Stresses in Terms of Displacements

In the previous section, equations relating strains and displacements along coordinates \(r, \theta, z\) were presented. In this section, using Hooke's law (equations 2.6), and the definitions of strains (equations 2.7), equations relating stresses and displacements are derived.

Using equations 2.6a and 2.7a the direct stress in the direction of \(r\) can be written as:

\[
\sigma_{rr} = \lambda \varepsilon + 2G \frac{\partial u}{\partial r} \quad (2.9a)
\]

Using equations 2.6b and 2.7b gives the direct stress in the direction of \(\theta\) as:
Also, the direct stress in the direction of \( z \) can be obtained from equations 2.6c and 2.7c as:

\[
\sigma_{zz} = \lambda \varepsilon + 2G \frac{\partial w}{\partial z} \quad (2.9c)
\]

Combining equations 2.6d, 2.6e, 2.6f with equations 2.7d, 2.7e and 2.7f respectively, the shear stresses in terms of displacements can be presented as:

\[
\tau_{r\theta} = G \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right) \quad (2.9d)
\]

\[
\tau_{rz} = G \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (2.9e)
\]

\[
\tau_{\theta z} = G \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \quad (2.9f)
\]

Where the volumetric strain \( \varepsilon \) is given by:

\[
\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta \theta} + \varepsilon_{zz} = \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \frac{\partial w}{\partial z} \right) \quad (2.10)
\]
2.6 Equations of Motion

In this section, by combining equations 2.2, 2.8 and 2.9, the equations of motion along the cylindrical coordinates are derived in terms of elastic rotations. In deriving the set of equations 2.2, the general body forces per unit volume in the direction of the cylindrical coordinates \( r, \theta, z \) were introduced as \( F_r, F_\theta \) and \( F_z \).

In general the term body forces can refer to gravitational, magnetic and inertial forces. In this investigation, only the inertial forces are considered. Hence, the body forces become:

\[
F_r = -\rho \frac{\alpha^2 u}{at^2} \quad \text{(2.11a)}
\]

\[
F_\theta = -\rho \frac{\alpha^2 v}{at^2} \quad \text{(2.11b)}
\]

\[
F_z = -\rho \frac{\alpha^2 w}{at^2} \quad \text{(2.11c)}
\]

where \( \rho = \) is the mass density of an isotropic and homogeneous material.

Differentiating equations 2.9a through 2.10 results in:

\[
\frac{\partial \sigma_{rr}}{\partial r} = \lambda \frac{\partial \varepsilon}{\partial r} + 2G \frac{\alpha^2 u}{at^2} \quad \text{(2.12a)}
\]
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial^2 v}{\partial r \partial \theta} - \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 w}{\partial r \partial \theta} - \frac{\partial^2 w}{\partial \theta^2}
\] (2.12d)

Combining equations 2.12, 2.11a with equations 2.2a, 2.9a and 2.9b gives:

\[
\frac{\partial^2 u}{\partial t^2} = \lambda \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial^2 v}{\partial r \partial \theta} - \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 w}{\partial r \partial \theta} - \frac{\partial^2 w}{\partial \theta^2} \right) + 2G \left( \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{2r} \frac{\partial^2 w}{\partial \theta \partial \phi} + \frac{1}{2r^2} \frac{\partial^2 w}{\partial \theta^2} \right)
\] (2.12e)

The above expression can be further simplified by using equation 2.12d:
or

\[ \frac{\partial^2 u}{\partial t^2} = (\lambda + 2G) \frac{\partial \varepsilon}{\partial r} + 2G(A + \frac{B}{r}) \]  \hspace{1cm} (2.13)

Where:

\[ 2A = \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial r \partial z} \]

\[ 2B = - \left( \frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} \right) \]

Differentiating equations 2.8b and 2.8c:

\[ \frac{\partial \varepsilon}{\partial z} = \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial r \partial z} \]

\[ \frac{\partial \varepsilon}{\partial \theta} = \frac{\partial^2 v}{\partial r \partial \theta} + \frac{\varepsilon}{r \partial \theta} - \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} \]

Comparing the above two expressions with the expressions for \( A \) and \( B \) gives:

\[ A = \frac{\partial \varepsilon}{\partial z} \]

and

\[ B = -\frac{\partial \varepsilon}{\partial \theta} \]

The equation of motion in the radial direction (2.13) can be written
as:

\[
\frac{\partial^2 u}{\partial t^2} = V_1^2 \frac{\partial^2 u}{\partial r^2} - \frac{2V_2^2}{r} \frac{\partial \omega_z}{\partial \theta} + 2V_2^2 \frac{\partial \omega_\theta}{\partial z} \quad (2.14)
\]

Where

\[
V_1^2 = \frac{\lambda + 2G}{\rho} \quad \text{velocity of propagation of dilatational waves}
\]

\[
V_2^2 = \frac{G}{\rho} \quad \text{velocity of propagation of distortional waves}
\]

The equation of motion in the \( \theta \) direction can be derived using a procedure similar to the one outlined above. Equations 2.9d, 2.9b, 2.9f and 2.10 give:

\[
\frac{\partial^2 \sigma_{r\theta}}{\partial r \partial \theta} = G \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial \theta} + \frac{v}{r^2} \right) \quad (2.15a)
\]

\[
\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = \lambda \left( \frac{\partial v}{\partial \theta} + \frac{2G}{\rho} \frac{\partial^2 v}{\partial \theta^2} + \frac{2G}{\rho} \frac{\partial u}{\partial \theta} \right) \quad (2.15b)
\]

\[
\frac{\partial \tau_{\theta z}}{\partial z} = G \left( \frac{\partial^2 v}{\partial z^2} + \frac{1}{r} \frac{\partial^2 w}{\partial z \partial \theta} \right) \quad (2.15c)
\]

\[
\frac{\partial \varepsilon}{\partial \theta} = \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial^2 w}{\partial z \partial \theta} \quad (2.15d)
\]
Combining equations 2.15a and through 2.15c and 2.11b with 2.2b gives:

\[
\rho \frac{\partial ^{2} v}{\partial t^{2}} = \frac{\lambda}{r} - \frac{\partial v}{\partial \theta} + \frac{2G}{r} \left( \frac{\partial ^{2} v}{\partial \theta ^{2}} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{r}{2} \frac{\partial ^{2} v}{\partial r ^{2}} \right)
\]

\[
+ \frac{1}{2} \frac{\partial ^{2} u}{\partial r \partial \theta} - \frac{1}{2r} \frac{\partial u}{\partial \theta} + \frac{\partial}{\partial r} \left( \frac{\partial v}{\partial r} - \frac{\partial u}{\partial \theta} \right) - \frac{1}{2} \frac{\partial ^{2} w}{\partial \theta ^{2}} + \frac{\partial v}{\partial \theta} + \frac{1}{2} \frac{\partial u}{\partial \theta}
\]

Using equation 2.15d

\[
\rho \frac{\partial ^{2} v}{\partial t^{2}} = \frac{\lambda}{r} - \frac{\partial v}{\partial \theta} + \frac{2G}{r} \left( \frac{\partial ^{2} v}{\partial \theta ^{2}} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial \theta} + \frac{r}{2} \frac{\partial ^{2} v}{\partial r ^{2}} + \frac{\partial v}{\partial r} \right)
\]

\[
+ \frac{1}{2} \frac{\partial ^{2} w}{\partial r \partial \theta} + \frac{\partial v}{\partial \theta} + \frac{1}{2} \frac{\partial u}{\partial \theta} - \frac{1}{2r} \frac{\partial u}{\partial \theta}
\]

\[
\rho \frac{\partial ^{2} v}{\partial t^{2}} = \frac{(\lambda + 2G)}{r} \frac{\partial v}{\partial \theta} + 2G \left( - \frac{1}{2} \frac{\partial ^{2} u}{\partial r \partial \theta} \right)
\]

\[
+ \frac{1}{2} \frac{\partial ^{2} w}{\partial r \partial \theta} + \frac{1}{2} \frac{\partial ^{2} v}{\partial r ^{2}} + \frac{1}{2r ^{2}} \frac{\partial u}{\partial \theta}
\]

\[
+ \frac{\partial ^{2} v}{\partial r ^{2}} - \frac{\partial v}{\partial r} + \frac{1}{2} \frac{\partial v}{\partial \theta}
\]

(2.16)
From equations 2.8a and 2.8c:

\[
\frac{\partial \omega}{\partial r} = \frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} - \frac{\partial \nu}{\partial r^2} - \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial u}{\partial \theta}
\]

\[
\frac{2 \omega}{\partial z} = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 \nu}{\partial z^2}
\]

Using the above two expressions, equation 2.16 can be written as:

\[
\frac{\partial^2 \nu}{\partial t^2} = \frac{\partial \nu}{\partial \theta} + 2V_2^2 \frac{\partial \omega}{\partial r} - 2V_2^2 \frac{\partial \omega}{\partial z} \quad (2.17)
\]

Equations 2.9e, 2.9f, 2.9c and 2.10 yield:

\[
\frac{\partial \tau_{rz}}{\partial r} = G \left( \frac{\partial^2 u}{\partial r \partial z} + \frac{\partial^2 w}{\partial r^2} \right) \quad (2.18a)
\]

\[
\frac{\partial \tau_{\theta z}}{\partial \theta} = G \left( \frac{\partial^2 v}{\partial z \partial \theta} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \quad (2.18b)
\]

\[
\frac{\partial \sigma_{zz}}{\partial z} = \lambda \frac{\partial \nu}{\partial z} + 2G \frac{\partial^2 w}{\partial z^2} \quad (2.18c)
\]

\[
\frac{\partial \nu}{\partial z} = \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial \nu}{\partial z \partial \theta} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial z} \quad (2.18d)
\]

Combining equations 2.18a, 2.18b, 2.18c, 2.9e and 2.11c with equation 2.2c results in:
\[
\rho \frac{\partial^2 w}{\partial t^2} = \lambda \frac{\partial^2 \varepsilon}{\partial z^2} + 2G \left( \frac{1}{2} \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{2} \frac{\partial^2 w}{\partial r^2} \right) \\
+ \frac{1}{2r} \frac{\partial^2 v}{\partial z \partial \theta} + \frac{1}{2r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} \\
+ \frac{1}{2r} \frac{\partial u}{\partial z} + \frac{1}{2r} \frac{\partial w}{\partial r} \tag{2.18e}
\]

Using equation 2.18d the term inside the parentheses in the above expression can be written as:

\[
2G \left( \frac{\partial^2 \varepsilon}{\partial z^2} - \frac{1}{2r} \frac{\partial^2 v}{\partial z \partial \theta} - \frac{1}{2r} \frac{\partial u}{\partial z} + \frac{1}{2r} \frac{\partial w}{\partial r} \right) \\
+ \frac{1}{2r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{2r} \frac{\partial^2 w}{\partial r \partial z} \tag{2.18e(a)}
\]

Using the above expression, equation 2.18e can be written as:

\[
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + 2G) \frac{\partial^2 \varepsilon}{\partial z^2} + \frac{2G}{\partial z} \left( \frac{\partial^2 \varepsilon}{\partial z \partial \theta} - \frac{1}{2r} \frac{\partial u}{\partial z} \right) \\
- \frac{1}{2} \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial w}{\partial r} + \frac{r}{2} \frac{\partial^2 w}{\partial r^2} \\
+ \frac{1}{2r} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{2r} \frac{\partial^2 u}{\partial r \partial z} \tag{2.18f}
\]
From equations 2.8a and 2.8b:

\[
2 \frac{\partial^2 u}{\partial t^2} = \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 v}{\partial z \partial \theta}
\]

\[
2 \frac{\partial^2 \theta}{\partial r \partial r} = \frac{\partial^2 u}{\partial z \partial r} + \frac{\partial u}{\partial z} - \frac{\partial^2 w}{\partial r^2} - \frac{\partial w}{\partial r}
\]

and equation 2.18f can be finally written as:

\[
\frac{\partial^2 w}{\partial t^2} = V_1^2 \frac{\partial^2 \epsilon}{\partial z} - 2V_2^2 \frac{\partial \omega_z}{\partial \theta} - 2V_2^2 \frac{\partial \omega_r}{\partial r} + \frac{\partial (\rho \omega_{\theta})}{\partial r}
\]

In summary, the equations of motion in cylindrical coordinates in terms of elastic rotations and displacements are:

\[
\frac{\partial^2 u}{\partial t^2} = V_1^2 \frac{\partial^2 \epsilon}{\partial \theta} - 2V_2^2 \frac{\partial \omega_z}{\partial \theta} + 2V_2^2 \frac{\partial \omega_{\theta}}{\partial z}
\] (2.19a)

\[
\frac{\partial^2 v}{\partial t^2} = V_1^2 \frac{\partial^2 \epsilon}{\partial r} + 2V_2^2 \frac{\partial \omega_z}{\partial r} - 2V_2^2 \frac{\partial \omega_r}{\partial z}
\] (2.19b)

\[
\frac{\partial^2 w}{\partial t^2} = V_1^2 \frac{\partial^2 \epsilon}{\partial z} + 2V_2^2 \frac{\partial \omega_r}{\partial \theta} - 2V_2^2 \frac{\partial \omega_{\theta}}{\partial r} + \frac{\partial (\rho \omega_{\theta})}{\partial r}
\] (2.19c)
CHAPTER THREE

FORCED VIBRATIONS OF CIRCULAR CYLINDERS

The purpose of this chapter is to study the forced vibrations of an infinitely long circular cylinder. In order to study the vibrations of the isotropic elastic cylinder shown in Figure 3.1, the governing equations described in the previous chapter are employed. The three equations of motion (eqs 2.19) are combined in such a way that four Bessel differential equations are obtained. The general theory of Bessel functions is used to solve these differential equations for the volumetric strain and the three elastic rotations. The equations relating displacements and rotations are used to obtain the displacements. Once the displacements are found, the six stress components for any point in the cylinder can be obtained. Hence, six equations for stresses and three equations for displacements are obtained in terms of Bessel functions. By considering an infinitely long cylinder subjected to an harmonic excitation, six boundary stresses need to be satisfied. By satisfying these six boundary conditions, the six stresses and the three displacements at any point in a cylinder can be computed for any given exciting frequency. As it turns out, the natural frequencies for the isotropic cylinder can be estimated by observing the values of exciting frequency which produce maximum amplitudes for stresses and displacements (resonance condition). The procedure summarized herein is described extensively in the pages to follow.
Figure 3.1 Dimensions and reference coordinates of an isotropic and elastic cylinder

3.1 Solution of Equations of Motion

As mentioned in the introductory chapter, and as became evident upon examining the equations of motion derived in Chapter Two, the study of vibrations of an elastic solid amounts to the solution of a wave propagation problem. Before attempting the solution of the equations of motion (eqs. 2.19), it is imperative to study the
types of waves considered in this investigation. In equations
2.19, the parameters $V_1$ and $V_2$ were introduced as the velocities of
propagation of the dilatational and distortional wave respectively.
Both of these velocities can be defined a ratio of the deformability
(elasticity) and inertia (density) of the medium. Whereas the inertia
of the medium resists the motion, the elasticity of the medium sus-
tains the motion.

As shown in Figure 3.2, dilatational (also called longitudinal)
waves are axially symmetric, and they are associated with dis-
placements in the radial and axial directions. In this investigation,
the wave propagation problem is simplified by considering an inﬁ-
initely long medium in the axial direction $z$ (infinite wavelength).
However, the technique employed herein can be used to obtain
approximate solutions to the wave propagation problem in a bounded
elastic medium (ﬁnite length hollow cylinder). In section 3.1.2, the
parameter $\gamma$ is introduced. Since $\gamma$ is a function of the cylinder
length ($L$), a ﬁnite length cylinder can be approximated by choosing
the appropriate value for the height to length ratio ($H/L$).

As shown in Figures 3.3 through 3.7, distortional (trans-
sverse) waves are associated with displacements in the tangential
direction $\theta$. In the present investigation, five different numbers of
distortional waves around the circumference are considered. In par-
ticular $n = 0, 1, 2, 3$ and $4$ corresponds to the axisymmetric motion,
rigid body motion, first, second and third lobar motions respectively.
Figure 3.2 Wave propagation in a cylinder

Figure 3.3 Axisymmetric vibration $n = 0$. 
Figure 3.4  Rigid body motion $n = 1$.

Figure 3.5  Nodal points for first lobar mode $n = 2$. 
Figure 3.6  Nodal points for second lobar mode \( n = 3 \).

Figure 3.7  Nodal points for third lobar mode \( n = 4 \).
3.1.1 Modified Equations of Motion

Since the dynamic response of a circular cylinder is governed by the equations of motion (eqs. 2.19), the vibration analysis of a cylinder requires the solution of these partial differential equations. However, because equations 2.19 do not allow a direct solution, they must be rewritten in a form from which a general solution can be obtained. In this subsection, equations 2.19 are rewritten as four modified equations by assuming harmonic displacements.

If harmonic displacements are assumed, the acceleration components in equations 2.19 become:

\[
\frac{a^2 u}{at^2} = -p^2 u
\]

\[
\frac{a^2 v}{at^2} = -p^2 v
\]

\[
\frac{a^2 w}{at^2} = -p^2 w
\]

and equations 2.19 can be written as:

\[
-pp^2 ru = (\lambda + 2G)r \frac{\partial \varepsilon}{\partial r} - 2G \frac{\partial \omega_z}{\partial \theta} + 2Gr \frac{\partial \omega_\theta}{\partial z} \quad (3.1)
\]

\[
-pp^2 v = (\lambda + 2G) \frac{1}{r} \frac{\partial \varepsilon}{\partial \theta} - 2G \frac{\partial \omega_r}{\partial z} + 2G \frac{\partial \omega_z}{\partial r} \quad (3.2)
\]
\[-p \rho^2 r w = (\lambda + 2G)r \frac{\partial \varepsilon}{\partial z} - 2G \frac{\partial (r \omega)}{\partial r} + 2G \frac{\partial \omega}{\partial \theta} \]  
(3.3)

Four important relations can be obtained from equations 2.8 (elastic rotations). These relations are very useful in the foregoing derivation and they are given by equations 3.4 below:

\[\frac{\partial \omega}{\partial r} + \frac{\partial \omega}{\partial \theta} + \frac{\partial (\omega \omega)}{\partial z} = 0 \]  
(3.4a)

\[\omega_r + r \frac{\partial \omega}{\partial r} + \frac{\partial \omega}{\partial \theta} + r \frac{\partial \omega}{\partial z} = 0 \]  
(3.4b)

\[\frac{\partial \omega}{\partial \theta} + r \frac{\partial^2 \omega}{\partial r \partial \theta} + \frac{\partial^2 \omega}{\partial \theta^2} + r \frac{\partial^2 \omega}{\partial z \partial \theta} = 0 \]  
(3.4c)

\[-r \frac{\partial^2 \omega}{\partial r \partial \theta} - r \frac{\partial^2 \omega}{\partial z \partial \theta} = \frac{\partial \omega}{\partial \theta} + \frac{\partial^2 \omega}{\partial \theta^2} \]  
(3.4d)

Differentiating equations 3.1, 3.2 and 3.3 with respect to r, \(\theta\), and z respectively results in:

\[-p \rho^2 (\frac{\partial (r \omega)}{\partial r} + \frac{\partial \omega}{\partial \theta} + \frac{\partial (r \omega)}{\partial z}) = (\lambda + 2G)(\frac{r \omega^2}{\partial z^2} + \frac{\partial \varepsilon}{\partial r} + \frac{1}{\partial r} \frac{\partial \varepsilon}{\partial \theta} + \frac{r \omega^2}{\partial z^2}) \]  
(3.5)

The expression in the last parenthesis can be shown to be equal to zero, and equation 3.5 can be further simplified as:
\[ r^2 \frac{\partial^2 \varepsilon}{\partial r^2} + r \frac{\partial \varepsilon}{\partial r} + r^2 \frac{\partial^2 \varepsilon}{\partial z^2} + \frac{\partial^2 \varepsilon}{\partial \theta^2} + \beta^2 r^2 \varepsilon = 0 \]  

(3.6)

where:

\[ \beta^2 = \frac{\rho P^2}{\lambda + 2G} \]  

(3.7)

Equations 3.1 and 3.2 can be rewritten as:

\[-\rho p^2 v r = (\lambda + 2G) \frac{\partial \varepsilon}{\partial \theta} - 2Gr \frac{\partial \omega_r}{\partial z} + 2Gr \frac{\partial \omega_z}{\partial r} \]  

(3.8a)

\[-\rho p^2 u = (\lambda + 2G) \frac{\partial \varepsilon}{\partial r} - \frac{2G}{r} \frac{\partial \omega_z}{\partial \theta} + 2G \frac{\partial \omega_\theta}{\partial z} \]  

(3.8b)

Subtracting the derivative of equation 3.8b with respect to \( \theta \) from the derivative of equation 3.8a with respect to \( r \) results in:

\[-\rho p^2 \left( \frac{\partial (vr)}{\partial r} - \frac{\partial u}{\partial \theta} \right) = 2G \left( - \frac{\partial^2 (r \omega_z)}{\partial r \partial z} + \frac{\partial^2 \omega_z}{\partial r^2} \frac{\partial \omega_z}{\partial r} + \frac{\partial^2 \omega_z}{\partial \theta^2} - \frac{\partial^2 \omega_\theta}{\partial z \partial \theta} \right) \]  

(3.8c)

Using equations 2.8c and 3.4a, equation 3.8c can be written as:

\[ r^2 \frac{\partial^2 \omega_z}{\partial r^2} + r \frac{\partial \omega_z}{\partial r} + r^2 \frac{\partial^2 \omega_z}{\partial z^2} + \frac{\partial^2 \omega_z}{\partial \theta^2} + \alpha^2 r^2 \omega_z = 0 \]  

(3.9)

where:

\[ \alpha^2 = \frac{\rho P^2}{G} \]  

(3.10)
From equations 3.2 and 3.3:

\[
\begin{align*}
-\frac{\rho p^2}{r} \frac{\partial w}{\partial \theta} &= \frac{(\lambda + 2G)}{r} - \frac{\partial^2 \varepsilon}{\partial z \partial \theta} - \frac{2G}{r^2} \frac{\partial^2 (r \omega_r)}{\partial r \partial \theta} + \frac{2G}{r^2} \frac{\partial^2 \omega_r}{\partial \theta^2} \\
\frac{-\rho p^2}{r} \frac{\partial v}{\partial z} &= \frac{(\lambda + 2G)}{r} - \frac{\partial^2 \varepsilon}{\partial z \partial \theta} - 2G \frac{\partial^2 \omega_r}{\partial z^2} + 2G \frac{\partial^2 \omega_z}{\partial r \partial z}
\end{align*}
\]  

(3.11a, 3.11b)

Subtracting 3.11b from 3.11a leads to:

\[
-\alpha^2 r^2 \omega_r = -\frac{\partial \omega_\theta}{\partial \theta} - \frac{\partial^2 \omega_\theta}{\partial r \partial \theta} + \frac{\partial^2 \omega_r}{\partial \theta^2} + \frac{\partial^2 \omega_r}{\partial z^2} - \frac{\partial^2 \omega_z}{\partial r \partial z} - \frac{\partial \omega_\theta}{\partial \theta}
\]

This expression can be simplified by using relation 3.4a as:

\[
-\alpha^2 r^2 \omega_r = \omega_r + \frac{\partial \omega_r}{\partial r} + \frac{\partial \omega_z}{\partial z} + \frac{\partial^2 \omega_r}{\partial \theta^2} + \frac{\partial^2 \omega_r}{\partial z^2} - \frac{\partial^2 \omega_z}{\partial r \partial z} - \frac{\partial \omega_\theta}{\partial \theta}
\]

Also, from 3.4a:

\[
-\frac{\partial}{\partial r} \left( \frac{\partial \omega_\theta}{\partial \theta} \right) = 2r \frac{\partial \omega_r}{\partial r} + \frac{\partial^2 \omega_r}{\partial r \partial \theta} + \frac{\partial \omega_z}{\partial z} + \frac{\partial^2 \omega_z}{\partial r \partial z}
\]

Combining the above two equations results in:

\[
\begin{align*}
\frac{\partial^2 \omega_r}{\partial r^2} + 3r \frac{\partial \omega_r}{\partial r} + (\alpha^2 r^2 + 1) \omega_r + \frac{\partial^2 \omega_r}{\partial \theta^2} + \frac{\partial^2 \omega_r}{\partial z^2} + 2 \frac{\partial \omega_z}{\partial z} &= 0
\end{align*}
\]  

(3.12)

Using a procedure similar to the one outlined above, another relation can be obtained from equations 3.1 and 3.3. Equation 3.13 below is obtained by differentiating equations 3.1 and 3.3 with respect to z.
and $r$ respectively and by using equations 3.4.

$$\omega_0 (r^2 \alpha^2 - 1) + r^2 \frac{\partial^2 \omega_\theta}{\partial r^2} + r \frac{\partial \omega_\theta}{\partial r} + r^2 \frac{\partial^2 \omega_\theta}{\partial z^2} + 2 \frac{\partial \omega_r}{\partial \theta} + \frac{\partial^2 \omega_\theta}{\partial \theta^2} = 0$$ (3.13)

Equations 3.6, 3.9, 3.12 and 3.13 are the modified equations of motion in terms four elastic functions (volumetric strain, and three elastic rotations).

### 3.1.2 Harmonic Motion

In this subsection, by assuming harmonic solutions, equations 3.6, 3.9, 3.12, and 3.13 are separated into four independent Bessel differential equations for the volumetric strain ($\varepsilon$) and the three elastic rotations ($\omega_r$, $\omega_\theta$ and $\omega_z$). The solutions to these four differential equations are readily obtained from the theory of Bessel functions.

The solutions to equations 3.6, 3.9, 3.12 and 3.13 are assumed to be of the form of equations 3.14 (Achenbach, 1973:239).

$$\varepsilon = E_r \cos(n\theta) \cos(pt + \gamma z)$$ (3.14a)

$$\omega_r = \Omega_r \sin(n\theta) \sin(pt + \gamma z)$$ (3.14b)

$$\omega_\theta = \Omega_\theta \cos(n\theta) \sin(pt + \gamma z)$$ (3.14c)

$$\omega_z = \Omega_z \sin(n\theta) \cos(pt + \gamma z)$$ (3.14d)
In the above equations, it is understood that $E_r$, $\Omega_r'$, $\Omega_\theta'$, $\Omega_z'$ are functions of $r$ only. Furthermore:

\[ p = \text{circular frequency} \]
\[ \tau = \text{wave number in axial direction} \]
\[ n = \text{number of circumferential waves} \]

Using equations 3.14a in equation 3.6 results in:

\[ r^2 \frac{\partial^2 E_r}{\partial r^2} + r \frac{\partial E_r}{\partial r} + (r^2\tau^2 - n^2)E_r = 0 \quad (3.15) \]

Similarly, by combining equation 3.14d with equation 3.9 results in:

\[ r^2 \frac{\partial^2 \Omega_z}{\partial r^2} + r \frac{\partial \Omega_z}{\partial r} + (\tau^2 r^2 - n^2)\Omega_z = 0 \quad (3.16) \]

In the above two equations:

\[ \tau^2 = \beta^2 - \chi^2 \quad (3.17a) \]
\[ \chi^2 = \alpha^2 - \chi^2 \quad (3.17b) \]

Using equation 3.14b and 3.14d, equation 3.12 can be written as:

\[ r^2 \frac{\partial^2 \Omega_\theta}{\partial r^2} + r \frac{\partial \Omega_\theta}{\partial r} + (\alpha^2 r^2 - r^2\tau^2 - 1 - n^2)\Omega_\theta + 2n\Omega_r = 0 \quad (3.18a) \]

Similarly equation 3.13 can be written as:

\[ r^2 \frac{\partial^2 \Omega_r}{\partial r^2} + 3r \frac{\partial \Omega_r}{\partial r} + (\alpha^2 r^2 - \chi^2 r^2 + 1 - n^2)\Omega_r - 2r\tau \Omega_z = 0 \quad (3.18b) \]
Adding equations 3.18a and 3.18b yields:

\[ (r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + (\alpha^2 r^2 - \gamma^2 r^2 - n^2))(\Omega_r + \Omega_\theta) + 2r \frac{\partial \Omega_r}{\partial r} + \Omega_r - \Omega_\theta - 2r \Omega_z + 2n \Omega_r = 0 \]  

(3.19)

Using equations 3.14b, 3.14c, 3.14d along with their appropriate derivatives, equation 3.4a yields:

\[ 2r \frac{\partial \Omega_r}{\partial r} = -2\Omega_r + 2n \Omega_\theta + 2r \Omega_z \]  

(3.20)

Using equation 3.21, equation 3.19 becomes:

\[ (r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + (\chi^2 r^2 - (n - 1)^2))(\Omega_r + \Omega_\theta) = 0 \]  

(3.21)

Subtracting equation 3.18b from 3.18a and using equation 3.20 gives:

\[ (r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + (\chi^2 r^2 - (n+1)^2))(\Omega_r - \Omega_\theta) = 0 \]  

(3.22)

Equations 3.15, 3.16, 3.21 and 3.22 are recognized as Bessel equations which solutions are given by:

\[ E_r = A_1 Z_n(\gamma_1 r) + A_4 W_n(\gamma_1 r) = f_1 \]  

(3.23a)

\[ \Omega_z = A_2 Z_n(\chi_1 r) + A_5 W_n(\chi_1 r) = f_2 \]  

(3.23b)
\[ \Omega_r + \Omega_\theta = 2A_2 Z_{n-1}(x_1 r) + 2A_3 W_{n-1}(x_1 r) = f_3 \quad (3.23c) \]

\[ \Omega_r - \Omega_\theta = 2A_3 Z_{n+1}(x_1 r) + 2A_6 W_{n+1}(x_1 r) = f_4 \quad (3.23d) \]

where \( \Psi_1 r = |\Psi r| \) and \( x_1 r = |x r| \)

In equations 3.23, \( Z_n \) and \( W_n \) represent the Bessel function of order \( n \) of the first and second kind \( (J_n \text{ and } Y_n) \), or the modified Bessel functions of order \( n \) of the first and second kind \( (I_n \text{ and } K_n) \) of the argument \( x r \) or \( \Psi r \). The choice of the appropriate Bessel functions depends on whether the argument is positive or negative. If the argument is positive, \( J_n \) and \( Y_n \) are used. On the other hand, if the argument is negative, \( I_n \) and \( K_n \) are used.

Noticing that \( V_1 \) is always greater than \( V_2 \), three different frequency intervals are to be considered in solving equations 3.15, 3.16, 3.21 and 3.22. Hence, three different sets of solutions are possible. In summary, these three sets are:

<table>
<thead>
<tr>
<th>Frequency Interval</th>
<th>Bessel functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 r &lt; p )</td>
<td>( J(\Psi_1 r), Y(\Psi_1 r), J(x_1 r), Y(x_1 r) )</td>
</tr>
<tr>
<td>( V_2 r &lt; p &lt; V_1 r )</td>
<td>( I(\Psi_1 r), K(\Psi_1 r), J(x_1 r), Y(x_1 r) )</td>
</tr>
<tr>
<td>( V_2 r &gt; p )</td>
<td>( I(\Psi_1 r), K(\Psi_1 r), I(x_1 r), K(x_1 r) )</td>
</tr>
</tbody>
</table>

Table 3.1 Bessel functions used at three frequency intervals

Since only three equations of motion (eqs 2.19) govern the dynamic response of a circular cylinder, it can be shown that any one of the three potentials \( f_2, f_3 \) and \( f_4 \) in equations 3.23 can be
set to zero without loss of the generality of the solution (Achenbach, 1973:239). Hence, $f_3$ can be set equal to zero by letting $\Omega_r = \Omega_\theta = f_4$. And by using equations 3.23, equations 3.14 can be written as:

$$\varepsilon = - \beta^2 f_1 \cos(n\theta) \cos(pt + az)$$  \hspace{1cm} (3.24a)

$$2\omega_z = \alpha^2 f_z \sin(n\theta) \cos(pt + az)$$  \hspace{1cm} (3.24b)

$$2\omega_\theta = - \alpha^2 f_4 \cos(n\theta) \sin(pt + az)$$  \hspace{1cm} (3.24c)

$$2\omega_r = \alpha^2 f_4 \sin(n\theta) \sin(pt + az)$$  \hspace{1cm} (3.24d)

Furthermore, by assuming that all displacements and stresses vary sinusoidally in phase at the same frequency, the time dependence can be omitted from equations 3.24, and they can be written as:

$$\varepsilon = - \beta^2 f_1 \cos(n\theta) \cos(\tau z)$$  \hspace{1cm} (3.25a)

$$2\omega_z = \alpha^2 f_z \sin(n\theta) \cos(\tau z)$$  \hspace{1cm} (3.25b)

$$2\omega_\theta = - \alpha^2 f_4 \cos(n\theta) \sin(\tau z)$$  \hspace{1cm} (3.25c)

$$2\omega_r = \alpha^2 f_4 \sin(n\theta) \cos(\tau z)$$  \hspace{1cm} (3.25d)

### 3.2.1 Displacements Equations

In this section, using the relations among stresses, displacements and elastic rotations developed in chapter two along with equations 3.25, the equations for displacements are derived in terms
of Bessel functions. From equations 3.1, 3.2 and 3.3, the three harmonic displacements can be written as:

\[ u = - \frac{1}{\beta^2} \frac{\partial x}{\partial r} + \frac{1}{r} \frac{\partial w_z}{\partial \theta} - \frac{2}{\alpha^2} \frac{\partial \omega}{\partial z} \]  
\[ (3.26a) \]

\[ v = - \frac{1}{r\beta^2} \frac{\partial x}{\partial \theta} - \frac{2}{\alpha^2} \frac{\partial \omega_z}{\partial r} + \frac{2}{\alpha^2} \frac{\partial \omega_r}{\partial z} \]  
\[ (3.26b) \]

\[ w = \frac{1}{\beta^2} \frac{\partial x}{\partial z} - \frac{2}{r\alpha^2} \frac{\partial \omega_r}{\partial \theta} + \frac{2}{r\alpha^2} \frac{\partial (r\omega_\theta)}{\partial r} \]  
\[ (3.26c) \]

Using equations 3.25 along with their appropriate derivatives, equations 3.26 can be rewritten as:

\[ u = \{ f'_1 + (n/r)f_2 + \gamma f_4 \} \cos(n\theta) \cos(\gamma z) \]  
\[ (3.27) \]

\[ v = \{ (-n/r)f_1 - f'_2 + \gamma f_4 \} \sin(n\theta) \cos(\gamma z) \]  
\[ (3.28) \]

\[ w = \{ -\gamma f_1 - f'_4 - ((n+1)/r)(f_4/r) \} \cos(n\theta) \sin(\gamma z) \]  
\[ (3.29) \]

By combining the above three equations with equations 3.23, the three harmonic displacements can be expressed as:

\[ u = \sum_{j} T_{7j} A_j \cos(n\theta) \cos(\gamma z) \quad j = 1, 2 \ldots 6 \]  
\[ (3.30) \]

\[ v = \sum_{j} T_{8j} A_j \sin(n\theta) \cos(\gamma z) \quad j = 1, 2 \ldots 6 \]  
\[ (3.31) \]

\[ w = \sum_{j} T_{9j} A_j \cos(n\theta) \sin(\gamma z) \quad j = 1, 2 \ldots 6 \]  
\[ (3.32) \]
3.2.2 Stress Equations

Equations 3.27, 3.28 and 3.29 can be combined with the stress-strain relations from the theory of elasticity to obtain the stresses in terms of equations 3.23. These relations between stresses and strains were derived in Chapter Two (eqs. 2.9).

Hence,

\[ \sigma_{rr} = \sum T_{1j} A_j \cos(n\theta) \cos(\gamma z) \quad j = 1, 2 \ldots 6 \]  \hspace{1cm} (3.33)

\[ \sigma_{\theta \theta} = \sum T_{2j} A_j \cos(n\theta) \cos(\gamma z) \quad j = 1, 2 \ldots 6 \]  \hspace{1cm} (3.34)

\[ \sigma_{zz} = \sum T_{3j} A_j \cos(n\theta) \cos(\gamma z) \quad j = 1, 2 \ldots 6 \]  \hspace{1cm} (3.35)

\[ \tau_{r\theta} = T_{4j} A_j \sin(n\theta) \cos(\gamma z) \quad j = 1, 2 \ldots 6 \]  \hspace{1cm} (3.36)

\[ \tau_{rz} = \sum T_{5j} A_j \cos(n\theta) \sin(\gamma z) \quad j = 1, 2 \ldots 6 \]  \hspace{1cm} (3.37)

\[ \tau_{\theta z} = \sum T_{6j} A_j \sin(n\theta) \sin(\gamma z) \quad j = 1, 2 \ldots 6 \]  \hspace{1cm} (3.38)

In equations 3.30 through 3.38, \( A_j \) is a constant of integration depending on the boundary conditions. The expressions for \( T_{ij} \) are given below:

\[ T_{11} = 2G \Psi_1^2 Z''_n (\Psi_1 r) - \lambda (\Psi^2 + \gamma^2) Z_n (\Psi r) \]

\[ T_{12} = \frac{2G_n}{r} \left( x_1 Z'_n(x_1 r) - \frac{Z_n(x_1 r)}{r} \right) \]

\[ T_{13} = 2G \gamma x_1 Z'_{n+1}(x_1 r) \]}
\[
T_{21} = \frac{2G\Psi_1}{r}Z_n'(\Psi_1 r) - \left(\lambda(\Psi^2 + \gamma^2) + \frac{2Gn^2}{r^2}\right)Z_n(\Psi_1 r)
\]

\[
T_{22} = \frac{2Gn}{r} \left(\frac{Z_n(x_1 r)}{r} - x_1 Z_n'(x_1 r)\right)
\]

\[
T_{23} = \left(\frac{2G\lambda(n+1)}{r}\right)Z_{n+1}(x_1 r)
\]

\[
T_{31} = -\left(\lambda(\Psi^2 + \gamma^2) + 2G\Psi^2\right)Z_n(\Psi_1 r)
\]

\[
T_{32} = T_{35} = 0
\]

\[
T_{33} = -2G\lambda \left(\frac{n+1}{r}\right)Z_{n+1}(x_1 r) + x_1 Z_{n+1}'(x_1 r)
\]

\[
T_{41} = \frac{2Gn}{r} \left(\frac{Z_n(\Psi_1 r)}{r} - \Psi_1 Z_n'(\Psi_1 r)\right)
\]

\[
T_{42} = G\left(\frac{x_1}{r} Z_n'(x_1 r) - Z_1^2 Z_n''(x_1 r) - \frac{n^2}{r^2} Z_n(x_1 r)\right)
\]

\[
T_{43} = G\lambda(x_1 Z_{n+1}'(x_1 r) - \frac{(n+1)}{r} Z_{n+1}(x_1 r))
\]

\[
T_{51} = -2G\lambda \Psi_1 Z_n'(\Psi_1 r)
\]

\[
T_{52} = -\frac{Gn\lambda}{r} Z_n(x_1 r)
\]
\[
T_{63} = \frac{G(n+1)}{r^2} Z_{n+1}^\prime(x_1r) - G \xi^2 Z_{n+1}^\prime(x_1r) - \frac{G(n+1)x_1}{r} Z_{n+1}^\prime(x_1r) G x_1^2 Z''_{n+1}(x_1r)
\]

\[
T_{61} = (2Gn\xi/r) Z_n(\psi_1r)
\]

\[
T_{62} = G\xi x_1 Z_n^\prime(x_1r)
\]

\[
T_{63} = \frac{Gn}{r} x_1 Z_{n+1}^\prime(x_1r) + \frac{G(n^2+n)}{r^2} Z_{n+1}^\prime(x_1r) - G\xi^2 Z_{n+1}^\prime(x_1r)
\]

\[
T_{71} = \psi_1 Z_n^\prime(\psi_1r)
\]

\[
T_{72} = (n/r) Z_n(x_1r)
\]

\[
T_{73} = \xi Z_{n+1}(x_1r)
\]

\[
T_{81} = -(n/r) Z_n(\psi_1r)
\]

\[
T_{82} = -x_1 Z_n^\prime(x_1r)
\]

\[
T_{83} = \xi Z_{n+1}(x_1r)
\]

\[
T_{91} = -\xi Z_n(\psi_1r)
\]

\[
T_{92} = T_{95} = 0
\]

\[
T_{93} = \frac{-(n+1)}{r} Z_{n+1}(x_1r) - x_1 Z_{n+1}^\prime(x_1r)
\]

In the above expressions the prime notation indicates differentiation with respect to \( r \).
The $T_{i4}$ terms are obtained by substituting $W_n', W'_n$ and $W''_n$ for $Z_n$, $Z'_n$ and $Z''_n$ respectively, in the terms $T_{i1}$. Similarly, the $T_{i5}$ terms are obtained by making the same substitutions in the $T_{i2}$ terms. Finally, the $T_{i6}$ terms are obtained by making similar substitutions in the $T_{i3}$ terms.

### 3.3 Stress Boundary Conditions

Considering an infinitely long cylinder with free ends, only six boundary conditions need to be satisfied. These boundary conditions are:

\[
\begin{align*}
\sigma_{rr}(a) &= P_0 \cos(n\theta) \cos \varphi z \\
\sigma_{rr}(b) &= 0 \\
\tau_{r\theta}(a) &= \tau_{r\theta}(b) = 0 \\
\tau_{rz}(a) &= \tau_{rz}(b) = 0
\end{align*}
\]

In matrix form, these boundary conditions can be represented as:

\[
\{\sigma\} = [S]\{A\} \tag{3.39}
\]

where:

- $\{\sigma\}$ is a 6x1 matrix
- $[S]$ is a 6x6 matrix
- $\{A\}$ is a 6x1 matrix
The first and second rows of \([S]\) are obtained from the \(T_{1j}\) terms by letting \(r=a\) and \(r=b\), respectively. The third and fourth rows of \([S]\) are determined from the \(T_{4j}\) terms by letting \(r=a\) and \(r=b\), respectively. Finally, the fifth and sixth rows of \([S]\) are obtained from the \(T_{5j}\) terms by letting \(r=a\) and \(r=b\), respectively.

### 3.4 Computations of Natural Frequencies

In matrix form, equations 3.30 through 3.38 can be written as:

\[
\begin{bmatrix}
\sigma_{rr} / \cos(n\theta) \cos(\gamma z) \\
\sigma_{\theta\theta} / \cos(n\theta) \cos(\gamma z) \\
\sigma_{zz} / \cos(n\theta) \cos(\gamma z) \\
\tau_{r\theta} / \sin(n\theta) \cos(\gamma z) \\
\tau_{r\gamma} / \cos(n\theta) \sin(\gamma z) \\
\tau_{\theta\gamma} / \sin(n\theta) \sin(\gamma z) \\
u / \cos(n\theta) \cos(\gamma z) \\
v / \sin(n\theta) \cos(\gamma z) \\
w / \cos(n\theta) \sin(\gamma z)
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\
T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\
T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\
T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\
T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\
T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \\
T_{71} & T_{72} & T_{73} & T_{74} & T_{75} & T_{76} \\
T_{81} & T_{82} & T_{83} & T_{84} & T_{85} & T_{86} \\
T_{91} & T_{92} & T_{93} & T_{94} & T_{95} & T_{96}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6
\end{bmatrix}
\]

or

\[\{U\} = [T]\{A\}\] (3.40)

where:

\(\{U\}\) is a 9x1 matrix.
\([T]\) is a 9x6 matrix

\((A)\) is a 6x1 matrix

Since matrices \([S]\) and \((\sigma)\) in equation 3.39 are known, the constants of integration can be found. Once the column matrix \((A)\) is known, the column matrix \((U)\) can be obtained from equation 3.40 for a given radius \(r\) and a given exciting frequency \(P\). The above procedure can be repeated for fixed valued of inside and outside radius, number of circumferential waves and cylinder length for a range of exciting frequencies. By observing the values of exciting frequency which produce the maximum amplitudes of stresses and displacements, the natural frequencies of the isotropic elastic cylinder can be determined. A computer can expedite the above procedure. The FORTRAN program described in the appendix is capable of performing this task.

The procedure described in this section was used to compute the natural frequencies for different cylinder geometries (\(H/R\) and \(H/L\)) and five different numbers of circumferential waves. The results obtained are presented and discussed in Chapter Four.
4.1 Material Properties and Dimensionless Parameters

Unless otherwise indicated, in this investigation the cylinder was assumed to be made of an elastic material having the following properties:

\[ G = 11538461 \, \text{PSI} \] \hspace{2cm} \text{Shear Modulus}

\[ \rho = 0.000725 \, \text{lb.s}^2/\text{in}^4 \] \hspace{2cm} \text{Mass Density}

\[ \nu = 0.30 \] \hspace{2cm} \text{Poisson's Ratio}

\[ n = 0.0 \] \hspace{2cm} \text{Material Damping Coefficient}

In order to present the results in a general form for future references and comparisons, the non-dimensional parameters listed below were utilized:

\[ \frac{H}{R} = \text{Cylinder's thickness to mean radius ratio} \]

\[ \frac{H}{L} = \text{Cylinder's thickness to half-wavelength ratio} \]

\[ \Omega = \frac{P}{\omega_s} \] \hspace{2cm} \text{Frequency factor}

\[ \omega_s = \frac{\pi V_2}{H} \] \hspace{2cm} \text{Fundamental frequency of an infinite plate of thickness H and same elastic properties as those of the cylinder}

\[ P = \text{Exciting frequency} \]

\[ V_2 = \text{Velocity of propagation of distortional waves} \]
4.2 Comparison with Gazis's Results

In order to verify the validity of the theoretical procedure used herein, the results for the resonant frequency factor ($\Omega$) obtained in this investigation are compared with the values of natural frequency factor presented by Gazis (1959). This comparison is presented in Tables 4.1 through 4.25. Five different values of cylinder's thickness to mean radius ratio are compared. For a given value of $H/R$, the frequency factors for five different numbers of circumferential waves are considered. For given values of $H/R$ and $n$, four different ratios of cylinder's thickness to half-wavelength ($H/L$) are compared. Since using a $H/L = 0.0$ results in a division by zero, a value of $H/L = 10x^{-20}$ is utilized.

A very small value of $H/L$ corresponds to an infinite wavelength (infinitely long cylinder). Similarly, a very small value of $H/R$ represents a thin cylindrical shell. On the other hand, $H/R = 2.00$ depicts a solid cylinder. Again, in order to avoid division by zero, $H/R = 2.00$ cannot be used, and a solid cylinder can be approximated by letting $H/R$ approach 2.00.

Upon examining Tables 4.1 through 4.25, it is evident that the two methods provide very similar results for the second and higher frequency factors for any values of $H/R$, $n$ and $H/L$. Furthermore, the two methods give similar results for the first frequency factor for any values of $H/R$, $n$ and $H/L$ up to 0.50. However, the two methods provide very dissimilar results for the first frequency factor at any $H/R$ and $n$ for $H/L = 1.00$. This dif-
ference seems to be inversely proportional to H/R.

It is important to recall that whereas Gazis's method uses a free vibrational approach, the present investigation uses a forced vibration approach. Both methods consider the vibration of an infinitely long circular cylinder using the three dimensional theory of elasticity. As a result, the frequency factors computed by Gazis (1959) should be very close to the resonant frequency factors obtained in this investigation. However, Tables 4.1 through 4.25 indicate that this is not always the case.

The difference in frequency factor mentioned above can be explained by considering the computational techniques for the Bessel functions values utilized in the two methods. Whereas Gazis (1959) uses an asymptotic approximation which is only valid for large values of the Bessel functions's argument, the present investigation utilizes a numerical technique to compute the values of the Bessel functions and their derivatives. This numerical technique is believed to provide accurate results for small values of the Bessel function's argument where the asymptotic approximation is not valid.

From equations 3.17a, 3.17b and Table 3.1, it is evident that the argument of the Bessel function (\( \gamma r \) or \( \beta r \)) is a function of cylinder's radius and wavelength as well as exciting frequency. For fixed values of cylinder's radius and wavelength, the argument is solely a function of exciting frequency. Indeed, as the exciting frequency increases, the argument (\( \gamma r \) or \( \beta r \)) increases, and the values of the Bessel functions obtained by asymptotic approximation approach the values obtained by the more accurate numerical
technique. This explains why the two methods provide very similar results for the second and higher modes.

Furthermore, from Tables 4.1 through 4.25, it is evident that for fixed values of H/L and n, the values of the frequency factors increase with increasing H/R. As a result, the argument of Bessel function increases, and the values of the Bessel function obtained by asymptotic approximation approach the value obtained by the numerical technique. Hence, as H/R increases the difference in frequency factor values for the two methods decreases.

4.3 Mode Identification

Tables 4.26 through 4.30 present the highest computed amplitude of displacements at the first six resonant frequencies for five different values of n for a cylinder with H/R = 0.30 and H/L = 0.10. These displacement amplitude are made dimensionless by multiplying them by factor G/P_oH, where G is the modulus of rigidity, P_o is the amplitude of the exciting pressure (σ_r) inside the cylinder, and H is the cylinder thickness.

It is noticed that theoretically these resonant displacement amplitude must be infinite since undamped motion is assumed (n = 0.0). From a mathematical point of view, at resonance condition (infinite displacements), the determinant of matrix |s| described in Chapter Three should be zero. Also, from linear algebra, it is recalled that a matrix |A| has an inverse in and only if its determinant is different than zero. The computer program listed in the
Appendix prints a "singular matrix" message whenever this condition is encountered. However, since the resonant frequencies computed in this investigation are approximations to the true natural frequencies of the cylinder, and because the computer could handle numbers as small as $10^{-20}$, a singular matrix was not encountered in the execution of the program.

The purpose of Tables 4.26 - 4.30 is the identification of the predominant vibrational mode for each value of $n$ and the above mentioned cylinder's dimensions. Indeed, by noticing which displacement component has the highest computed amplitude, the predominant mode of vibration can be identified.

In particular, Table 4.26 allows the identification of the predominant vibrational mode for axisymmetric vibration ($n = 0$). In this table, it is noticed that the non-dimensional tangential displacement, $v$, is zero. This comes as no surprise since the condition $n = 0$ does not allow motion in the $\theta$ direction (see Figure 3.4). Also from Table 4.26, it is evident that the first, third, fifth and sixth modes for $n = 0$ are associated with large $z$-displacements, and the second and fourth mode correspond to predominantly large radial displacements.

Table 4.27 depicts the mode identification for rigid body motion ($n = 1$). From this table, it can be concluded that the first and fifth modes for $n = 1$ are associated with displacements in the tangential direction. Furthermore, the third and sixth modes correspond to predominantly radial displacements. Similarly, the second and fourth modes are associated with displacements in the
Tables 4.28, 4.29 and 4.30 depict the mode identification for the first three lobar modes (n = 2, 3 and 4). On the basis of these tables, it can be concluded that the first and sixth mode for n = 2, 3 and 4 are associated with large radial displacements. The second and fourth mode are associated with displacement in the z-direction. Finally, the third and fifth mode are associated with displacements in the tangential direction.

4.4 Effect of Structural Damping

Tables 4.31 to 4.35 depict the variation of the fundamental resonant displacement amplitude with respect to structural (material) damping factor for H/R = 0.40, H/L = 0.10, ν = 0.30 and n ranging from 0 to 4. The displacements are made dimensionless as indicated in the previous section. In order to perform the structural damping analysis, the computer program in Appendix A was used, assuming a complex shear modulus of the form \( G^* = G(1 + i\eta) \). The symbol \( \eta \) represents the structural damping factor. Although they are not listed in Tables 4.31 to 4.35, the resonant frequencies for cylinders with material damping are the same as those of cylinders with no material damping. In other words, the natural frequencies of a system are independent of the damping present in the system. However, the resonant amplitude are damping dependent.

In particular, Table 4.31 presents the variation of the fundamental resonant displacements amplitude for axisymmetric vibrations.
(n = 0). As indicated in the previous section, the tangential displacement is zero for axisymmetric vibrations. Upon examining Table 4.31, it seems that the structural damping has no effect in the displacement amplitude for axisymmetric vibrations. The explanation to this lay in the fact that no shear deformation \((v = 0)\) is present for \(n = 0\), and as a result, energy dissipation occurs only because of displacements in the radial and axial directions. As shown by Hamidzadeh (1981, 1982), energy dissipation due to shear deformations is more effective in damping treatment than energy dissipation due to compressive or tensile deformations.

Since structural damping is not effective for axisymmetric vibration, the displacements amplitude listed in Table 4.31 should be infinite values. However, as indicated before, the tabulated values are only the highest possible computed displacement amplitude. The differences among them can be attributed to round off errors due to matrix inversions. Furthermore, for undamped motions, the displacement amplitude change very rapidly close to the resonant frequencies, causing a very sharp frequency response curve at resonance.

Tables 4.32 to 4.35 show the variation of the fundamental resonant displacement amplitude with respect to material damping for rigid body motion \((n = 1)\), and the first three lobar modes \((n = 2, 3, \text{ and } 4)\). Based on these tables, it can be concluded that the resonant displacement amplitude decrease as the material damping factor increases. This is so because of the presence of energy dissipation due to shear deformations as well as radial and axial deformations.
Figures 4.26 to 4.30 are graphic representations of Tables 4.31 to 4.35. From Figures 4.27 to 4.30, it is evident that the displacement amplitude decrease linearly with increasing material damping factor. This finding is consistent with previous theoretical and experimental results in structural damping. It is a well known fact that the steady state response amplitude of a system subjected to a harmonic excitation is given by:

\[
X = \frac{F_0}{(k - m\omega^2 + i\eta k)} \tag{4.1}
\]

where

- \(F_0\) = amplitude of harmonic excitation
- \(k\) = system's stiffness
- \(m\) = total effective mass
- \(\omega\) = frequency of excitation

From equation 4.1, it can be easily seen that for undamped motion (\(\eta = 0.0\)), the amplitude at resonance \((k - m\omega^2 = 0)\) is infinite. Also, for damped motion (\(\eta \neq 0\)), the resonant amplitude for fixed values of \(F_0\) and \(k\) is a linear function of \(\eta\). Indeed

\[|X| = \frac{F_0}{\eta k} \quad \text{(Thomson, 1981:75)}.
\]

4.5 Effect of Poisson's Ratio

The variation of the first resonant frequency with respect to Poisson's ratio is presented in Tables 4.36 to 4.40. This variation is studied using \(H/R = 1.00\) at three different values of \(H/L\) (0.10,
0.50 and 1.00) and \( n \) ranging from 0 to 4. The Poisson's ratio ranges between the two ideal values of 0 and 0.50.

Since the Poisson's ratio is defined as the lateral strain divided by the axial strain, an ideal material having a Poisson's ratio equal to zero could be stretched in one direction without any lateral deformation. On the other hand, an ideal material with a Poisson's ratio equal to 0.50 would be perfectly incompressible, since its bulk modulus (modulus of compression) would be infinite.

The results listed in Tables 4.36 to 4.40 were obtained by letting \( \nu \) (\( \nu_u \)) vary in the program listed in Appendix A. An exact value of \( \nu = 0.50 \) could not be used for computations. Instead, a value of 0.495 was used to approximate 0.50.

Based on Tables 4.36 to 4.40, it can be concluded that the value of the resonant frequency factor increases for increasing values of Poisson's ratio at any values of \( n \) and \( H/L \). This is explained by noticing that as the Poisson's approaches the ideal value of 0.50 the bulk modules of the material approaches infinity, and as a result the stiffness of the material increases. From basic vibrations knowledge, it is recalled that the natural frequency of an elastic system is directly proportional to the system's stiffness, and inversely proportional to the system's mass (density). Since in this analysis, the density of the material is held constant, the resonant frequency increases for increasing values of Poisson's ratio.

Upon examining table 4.36, it is noticed that the resonant frequency factor for \( n = 0 \) and \( H/L = 0.10 \) seems to decrease for increasing Poisson's ratio. Indeed, the resonant frequency factor at
\( \nu = 0.00 \) is three times higher than at \( \nu = 0.10 \). In section 4.4, it was pointed out that for axisymmetric vibration \((n = 0)\), the fundamental mode was associated with displacements in the \( z \)-direction. Moreover, an ideal material having a Poisson's ratio equal to zero would not experience any axial deformation, when subjected to radial excitation. As a result, the value listed in Table 4.36 for \( H/L = 0.10 \) and \( \nu = 0.0 \) corresponds to the second resonant frequency factor rather than the fundamental resonant frequency factor.

**4.6 Resonant Frequencies Charts**

Figures 4.1 through 4.25 graphically depict the resonant frequencies factors obtained in this investigation. These values are tabulated in Tables 4.1 through 4.25. Five different thickness to mean radius ratios are presented, \( H/R = 0.30, 0.40, 0.50, 1.00 \) and 1.90. For a given \( H/R \) value, \( H/L \) ratios ranging from 0 to 1 are included. Furthermore, for fixed values of \( H/R \) and \( H/L \), five different numbers of circumferential waves are provided, \( n = 0, 1, 2, 3, \) and 4. The first six resonant frequencies factors are provided for each of the above conditions.

These charts along with Tables 4.1 to 4.25 can be used as references in future investigations and in engineering design. Indeed, they provide a quick way of determining the natural frequencies of cylindrical structures for different lobar modes.

In order to determine a cylinder's natural frequencies from Figures 4.1 through 4.25, one needs to know the cylinder's
dimensions and elastic constants. The cylinder's dimensions provide the H/R and H/L ratios. Knowing the values of H/R and H/L, the frequency factor ($\Omega$) for a given value of $n$ can be read from the appropriate table or figure. On the other hand, knowing the cylinder's elastic (G and P) constants allows the calculation of $\omega_s = \pi V_2/H$. Finally, the natural frequency $P$ can be obtained from the relation $\Omega = P/\omega_s$.

When the natural frequencies for values of H/R not provided in Tables 4.1 through 4.25 and in Figures 4.1 through 4.25 are desired, one can interpolate between the appropriate tables or figures. However, for more accurate results, one must use the computer program listed in the appendix. Moreover, Figures 4.1 through 4.25 can be used to draw very important conclusions regarding the vibrations of circular structures. These conclusions are presented in the pages to follow.

4.7 Coupled Motion

As shown in figures 4.1, 4.6, 4.11, 4.16 and 4.21, there is a noticeable coupling effect between the first and second mode for axisymmetric vibration ($n = 0$). This mode coupling depends on the ratio H/L as well as the ratio H/R.

Indeed, the coupling is most noticeable for H/L values between 0.10 and 0.40. As H/L approaches zero (infinitely long cylinder), the axial wave number ($\bar{\tau}$) becomes zero, resulting in uncoupled radial or axial motion. On the other hand, for fixed values of
H/L, the coupling effect between the first two modes weakens as the ratio H/R increases.

Furthermore, as the coupling between the first and second mode decreases with increasing H/R, a coupling effect between the second and third mode is observed. However, the fourth mode always occurs at a frequency almost twice as high as the frequency of the third mode for H/L < 0.50. Hence, it can be concluded that there is a considerable coupling effect among the first three modes for any H/R at H/L < 0.50. This observation is consistent with the "three-mode theory" proposed by McNiven, Shah and Sackman in 1966.

The coupling among the first three modes is also observed at values of n = 1, 2, 3 and 4, at H/R values less than 1.00. This coupling is strongest at n = 1 (rigid body motion). However, as H/R becomes bigger than 1.00, approaching a solid rod (H/R = 2.00), the coupling effect among the first three modes weakens. Moreover, the first six modes move away from each other as H/R approaches 2.00 and n > 0 (see Tables 4.21 to 4.25). In fact, the frequency curves become almost parallel straight lines, as shown by Figure 4.23.
Table 4.1 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 0.30$ and $n = 0$ (symmetric radial vibration)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Present</th>
<th>Gazis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0</td>
<td>0.16328</td>
</tr>
<tr>
<td></td>
<td>1.00288</td>
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</tr>
<tr>
<td></td>
<td>2.00150</td>
<td>2.00174</td>
</tr>
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<td>0.19024</td>
</tr>
<tr>
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<td>2.03137</td>
</tr>
<tr>
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<tr>
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<td>2.33670</td>
<td>2.33665</td>
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<tr>
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<td>0.78475</td>
<td>1.41980</td>
</tr>
<tr>
<td></td>
<td>2.82845</td>
<td>2.82864</td>
</tr>
</tbody>
</table>

Table 4.2 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 0.30$ and $n = 1$ (rigid body motion)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Present</th>
<th>Gazis</th>
</tr>
</thead>
<tbody>
<tr>
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Table 4.3 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 0.30$ and $n = 2$ (first lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td></td>
<td>0.00</td>
</tr>
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</table>

Table 4.4 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 0.30$ and $n = 3$ (second lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>0.09512</td>
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<td>0.92900</td>
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</table>
Table 4.5 Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.30 and n = 4 (third lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Resonant Frequency Factors Ω</th>
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</table>

Table 4.6 Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.40 and n = 0 (symmetric radial vibration)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Resonant Frequency Factors Ω</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td>0.00</td>
<td>0 0.21878 1.00550 1.88498 2.00224 3.00150</td>
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<td>1.07641 1.42038 1.91490 2.00366 2.83127 3.08008</td>
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</table>
Table 4.7 Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.40 and n = 1 (rigid body motion)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Resonant Frequency Factors Ω</th>
</tr>
</thead>
<tbody>
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<td></td>
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Table 4.8 Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.40 and n = 2 (first lobar mode)

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<th>Thickness to length ratio H/L</th>
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Table 4.9  Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.40 and n = 3 (second lobar mode)

<table>
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<th>Thickness to length ratio H/R</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>Present 0.15855 0.38306 0.63729 1.08210 1.27294 1.76290</td>
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<tr>
<td></td>
<td>Gazis 0.14935 0.38306 0.63714 1.08227 1.27357 1.76321</td>
</tr>
<tr>
<td></td>
<td>Present 0.18390 0.40109 0.65631 1.08906 1.28404 1.75988</td>
</tr>
<tr>
<td></td>
<td>Gazis 0.16774 0.40061 0.65601 1.08854 1.28391 1.76045</td>
</tr>
<tr>
<td></td>
<td>Present 0.55328 0.67375 1.00347 1.21588 1.50912 1.75480</td>
</tr>
<tr>
<td></td>
<td>Gazis 0.39794 0.64374 1.00342 1.21630 1.50880 1.75441</td>
</tr>
<tr>
<td></td>
<td>Present 0.92107 1.10495 1.45367 1.54719 1.98782 2.10828</td>
</tr>
<tr>
<td></td>
<td>Gazis 0.84781 1.07579 1.43510 1.52367 1.98837 2.10841</td>
</tr>
</tbody>
</table>

Table 4.10  Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.40 and n = 4 (third lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/R</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present 0.27426 0.50916 0.81510 1.13837 1.41877 1.74687</td>
</tr>
<tr>
<td></td>
<td>Gazis 0.27416 0.50922 0.81561 1.13847 1.41926 1.74701</td>
</tr>
<tr>
<td></td>
<td>Present 0.28100 0.52157 0.83068 1.14385 1.42828 1.74687</td>
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<tr>
<td></td>
<td>Gazis 0.28134 0.52136 0.83022 1.14419 1.42795 1.74684</td>
</tr>
<tr>
<td></td>
<td>Present 0.65791 0.74192 1.10176 1.27137 1.61374 1.78650</td>
</tr>
<tr>
<td></td>
<td>Gazis 0.46980 0.72697 1.10200 1.27137 1.61323 1.78678</td>
</tr>
<tr>
<td></td>
<td>Present 0.88462 1.00351 1.14618 1.50282 2.03696 2.19071</td>
</tr>
<tr>
<td></td>
<td>Gazis 0.89257 1.12897 1.46091 1.56878 2.03657 2.19128</td>
</tr>
</tbody>
</table>
Table 4.11 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 0.50$ and $n = 0$ (symmetric radial vibration)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
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<td>0.0 0.27744 1.00840 1.89448 2.00325 3.00284</td>
</tr>
<tr>
<td>0.10</td>
<td>0.15061 0.28219 1.02724 1.87068 2.03710 3.00229</td>
</tr>
<tr>
<td>0.50</td>
<td>0.56596 0.80374 1.34428 1.75321 2.34304 3.00392</td>
</tr>
<tr>
<td>1.00</td>
<td>1.07875 1.42346 1.91977 2.00536 2.83426 3.08170</td>
</tr>
</tbody>
</table>

Table 4.12 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 0.50$ and $n = 1$ (rigid body motion)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td>0.00</td>
<td>0 0.16110 0.36545 1.02375 1.10016 1.84531</td>
</tr>
<tr>
<td>0.10</td>
<td>0.09354 0.21878 0.38365 1.03517 1.10967 1.83420</td>
</tr>
<tr>
<td>0.50</td>
<td>0.50413 0.59449 0.84654 1.17280 1.37950 1.75163</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00666 1.09226 1.41880 1.47586 1.93392 2.02427</td>
</tr>
</tbody>
</table>
Table 4.13 Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.50 and n = 2 (first lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Resonant Frequency Factors Ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>Gazis</td>
</tr>
<tr>
<td>0.00</td>
<td>0.09374 0.32038 0.55485 1.06525 1.23807 1.78507</td>
</tr>
<tr>
<td>0.10</td>
<td>0.10100 0.34560 0.57388 1.07321 1.24758 1.78032</td>
</tr>
<tr>
<td>0.50</td>
<td>0.54218 0.65632 0.95433 1.20954 1.47900 1.75321</td>
</tr>
<tr>
<td>1.00</td>
<td>1.02886 1.13031 1.42989 1.52658 1.97038 2.08291</td>
</tr>
</tbody>
</table>

Table 4.14 Comparison between the present results and Gazis's results for six frequency factors for H/R = 0.50 and n = 3 (second lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Resonant Frequency Factors Ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>Gazis</td>
</tr>
<tr>
<td>0.00</td>
<td>0.23146 0.47793 0.76412 1.13175 1.42045 1.75283</td>
</tr>
<tr>
<td>0.10</td>
<td>0.23780 0.49145 0.77998 1.13824 1.42996 1.75334</td>
</tr>
<tr>
<td>0.50</td>
<td>0.59926 0.73876 1.06850 1.27141 1.61069 1.78189</td>
</tr>
<tr>
<td>1.00</td>
<td>1.02412 1.15095 1.47750 1.60432 2.01971 2.18139</td>
</tr>
</tbody>
</table>
Table 4.15 Comparison between the present results and Gazis's results for six frequency factors for \( H/R = 0.50 \) and \( n = 4 \) (third lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio ( H/L )</th>
<th>Resonant Frequency Factors ( \Omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>Gazis</td>
</tr>
<tr>
<td>0.00</td>
<td>0.38841 0.63273 0.96860 1.22018 1.59947 1.78650</td>
</tr>
<tr>
<td>0.10</td>
<td>0.38836 0.63276 0.96798 1.22007 1.59990 1.78581</td>
</tr>
<tr>
<td>0.50</td>
<td>0.39477 0.64296 0.97970 1.22550 1.60581 1.78930</td>
</tr>
<tr>
<td>1.00</td>
<td>0.39397 0.64279 0.97921 1.22574 1.60519 1.78891</td>
</tr>
</tbody>
</table>

Table 4.16 Comparison between the present results and Gazis's results for six frequency factors for \( H/R = 1.00 \) and \( n = 0 \) (symmetric radial vibration)

<table>
<thead>
<tr>
<th>Thickness to length ratio ( H/L )</th>
<th>Resonant Frequency Factors ( \Omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>Gazis</td>
</tr>
<tr>
<td>0.00</td>
<td>0.60085 1.04285 1.99878 2.02470 3.01585</td>
</tr>
<tr>
<td>0.10</td>
<td>0.60059 1.04127 1.99847 2.02371 3.01636</td>
</tr>
<tr>
<td>0.50</td>
<td>0.59764 1.06136 1.95235 2.07671 3.01669</td>
</tr>
<tr>
<td>1.00</td>
<td>0.59789 1.06120 1.95264 2.07688 3.01611</td>
</tr>
</tbody>
</table>

| Present                           | Gazis                             |
| 0.63333 0.86158 1.39585 1.81655 2.39285 3.01836 |
| 0.55369 0.87513 1.39566 1.81691 2.39297 3.01856 |
| 1.11286 1.46795 1.97691 2.02365 2.87646 3.09852 |
| 0.89200 1.49809 1.97696 2.02388 2.87635 3.09887 |
Table 4.17 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.00$ and $n = 1$ (rigid body motion)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td>0.00</td>
<td>0.3263 0.62698 1.11853 1.46317 1.83632</td>
</tr>
<tr>
<td>0.10</td>
<td>0.36461 0.63806 1.12790 1.47023 1.83174</td>
</tr>
<tr>
<td>0.50</td>
<td>0.71178 0.94874 1.30154 1.64463 1.79831</td>
</tr>
<tr>
<td>1.00</td>
<td>1.16988 1.43613 1.65424 2.00041 2.14141</td>
</tr>
</tbody>
</table>

Table 4.18 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.00$ and $n = 2$ (first lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td>0.00</td>
<td>0.31066 0.62215 0.89094 1.33115 1.72803 1.97095</td>
</tr>
<tr>
<td>0.10</td>
<td>0.31290 0.63730 0.90350 1.33486 1.73100 1.97527</td>
</tr>
<tr>
<td>0.50</td>
<td>0.86241 1.13511 1.46483 1.77870 2.07354</td>
</tr>
<tr>
<td>1.00</td>
<td>1.28728 1.50287 1.81200 2.06410 2.35503</td>
</tr>
</tbody>
</table>
Table 4.19 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.00$ and $n = 3$ (second lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Present</th>
<th>Gazis</th>
<th>Present</th>
<th>Gazis</th>
<th>Present</th>
<th>Gazis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.65871</td>
<td>0.65849</td>
<td>0.88350</td>
<td>0.88365</td>
<td>1.21742</td>
<td>1.21733</td>
</tr>
<tr>
<td>0.10</td>
<td>0.65950</td>
<td>0.65957</td>
<td>0.89329</td>
<td>0.89336</td>
<td>1.22534</td>
<td>1.22510</td>
</tr>
<tr>
<td>0.50</td>
<td>0.80692</td>
<td>0.74090</td>
<td>1.05814</td>
<td>1.05812</td>
<td>1.37829</td>
<td>1.37817</td>
</tr>
<tr>
<td>1.00</td>
<td>1.19373</td>
<td>1.06030</td>
<td>1.43225</td>
<td>1.37805</td>
<td>1.65886</td>
<td>1.62382</td>
</tr>
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</table>

Table 4.20 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.00$ and $n = 4$ (third lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio H/L</th>
<th>Present</th>
<th>Gazis</th>
<th>Present</th>
<th>Gazis</th>
<th>Present</th>
<th>Gazis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.95912</td>
<td>0.95897</td>
<td>1.12685</td>
<td>1.12632</td>
<td>1.55504</td>
<td>1.55537</td>
</tr>
<tr>
<td>0.10</td>
<td>0.95978</td>
<td>0.95948</td>
<td>1.13347</td>
<td>1.13385</td>
<td>1.56058</td>
<td>1.56058</td>
</tr>
<tr>
<td>0.50</td>
<td>1.00984</td>
<td>1.00961</td>
<td>1.27057</td>
<td>1.27067</td>
<td>1.66517</td>
<td>1.66535</td>
</tr>
<tr>
<td>1.00</td>
<td>1.36170</td>
<td>1.24734</td>
<td>1.58752</td>
<td>1.55401</td>
<td>1.86743</td>
<td>1.86716</td>
</tr>
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</table>
Table 4.21 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.90$ and $n = 0$ (symmetric radial vibration)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
<th>Present</th>
<th>Gazis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>0.00</td>
<td></td>
<td>1.19120</td>
<td>1.19078</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.22893</td>
<td>1.22956</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.18415</td>
<td>2.18355</td>
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<td>3.11847</td>
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<td>3.16880</td>
<td>3.17090</td>
</tr>
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<td></td>
<td>0.15422</td>
<td>0.16086</td>
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<td>1.15486</td>
<td>1.15497</td>
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<td>1.27831</td>
<td>1.27838</td>
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<td></td>
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<td>2.18575</td>
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<td>3.11777</td>
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<td>0.73570</td>
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<td>2.52778</td>
<td>2.52757</td>
</tr>
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<td></td>
<td></td>
<td>3.25911</td>
<td>3.25915</td>
</tr>
<tr>
<td></td>
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<td>3.66574</td>
<td>3.66578</td>
</tr>
</tbody>
</table>

Table 4.22 Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.90$ and $n = 1$ (rigid body motion)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
<th>Present</th>
<th>Gazis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>0.00</td>
<td></td>
<td>0.57105</td>
<td>0.57205</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.87509</td>
<td>0.87415</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.64893</td>
<td>1.64883</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.99674</td>
<td>1.99648</td>
</tr>
<tr>
<td></td>
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<td>2.20271</td>
<td>2.20236</td>
</tr>
<tr>
<td>0.10</td>
<td></td>
<td>0.02457</td>
<td>0.02482</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.59529</td>
<td>0.59561</td>
</tr>
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<td>0.88167</td>
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<td>1.64875</td>
<td>1.64920</td>
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<tr>
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<td></td>
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<td>2.00173</td>
</tr>
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</tr>
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<td>2.09668</td>
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<td>2.33616</td>
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<td>0.87963</td>
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<td>1.25785</td>
<td>1.24776</td>
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<tr>
<td></td>
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<td>1.52666</td>
<td>1.54558</td>
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<tr>
<td></td>
<td></td>
<td>2.01180</td>
<td>2.01168</td>
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<td>2.29549</td>
<td>2.29528</td>
</tr>
<tr>
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<td></td>
<td>2.62997</td>
<td>2.62982</td>
</tr>
</tbody>
</table>
Table 4.23  Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.90$ and $n = 2$ (first lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td>0.00</td>
<td>0.72485 0.94775 1.35864 2.07120 2.42547 2.93287</td>
</tr>
<tr>
<td>0.10</td>
<td>0.72133 0.96070 1.36340 2.07997 2.43023 2.93599</td>
</tr>
<tr>
<td>0.50</td>
<td>0.72743 1.16364 1.53139 2.12753 2.49521 2.99154</td>
</tr>
<tr>
<td>1.00</td>
<td>0.99560 1.48704 1.90876 2.36362 2.67131 3.13567</td>
</tr>
</tbody>
</table>

Table 4.24  Comparison between the present results and Gazis's results for six frequency factors for $H/R = 1.90$ and $n = 3$ (second lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio $H/L$</th>
<th>Resonant Frequency Factors $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
</tr>
<tr>
<td>0.00</td>
<td>1.12090 1.30150 1.86430 2.48615 2.86948 3.51903</td>
</tr>
<tr>
<td>0.10</td>
<td>1.12084 1.31185 1.86911 2.48505 2.87264 3.51921</td>
</tr>
<tr>
<td>0.50</td>
<td>1.12915 1.45529 1.99907 2.53021 2.92484 3.50824</td>
</tr>
<tr>
<td>1.00</td>
<td>1.29795 1.76441 2.30667 2.72983 3.08186 3.59213</td>
</tr>
</tbody>
</table>
Table 4.25  Comparison between the present results and Gazis's results for six frequency factors for \( H/R = 1.90 \) and \( n = 4 \) (third lobar mode)

<table>
<thead>
<tr>
<th>Thickness to length ratio ( H/L )</th>
<th>Resonant Frequency Factors ( \Omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>Gazis</td>
</tr>
<tr>
<td>( 0.00 )</td>
<td>( 0.00 )</td>
</tr>
<tr>
<td><strong>H/L</strong></td>
<td><strong>H/L</strong></td>
</tr>
<tr>
<td>1.46326</td>
<td>1.46237</td>
</tr>
<tr>
<td>1.64915</td>
<td>1.64924</td>
</tr>
<tr>
<td>2.36525</td>
<td>2.36762</td>
</tr>
<tr>
<td>2.87775</td>
<td>2.87892</td>
</tr>
<tr>
<td>3.28515</td>
<td>3.28904</td>
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<td>3.93937</td>
<td>3.93327</td>
</tr>
<tr>
<td>1.46387</td>
<td>1.46266</td>
</tr>
<tr>
<td>1.65506</td>
<td>1.65521</td>
</tr>
<tr>
<td>2.37287</td>
<td>2.37186</td>
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<td>2.88085</td>
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<td>3.28626</td>
<td>3.29102</td>
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<tr>
<td>3.93462</td>
<td>3.93377</td>
</tr>
<tr>
<td>1.47595</td>
<td>1.47586</td>
</tr>
<tr>
<td>1.76559</td>
<td>1.76909</td>
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<tr>
<td>2.46986</td>
<td>2.46817</td>
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<tr>
<td>2.92811</td>
<td>2.92969</td>
</tr>
<tr>
<td>3.33540</td>
<td>3.33795</td>
</tr>
<tr>
<td>3.94732</td>
<td>3.94973</td>
</tr>
<tr>
<td>1.60912</td>
<td>1.61050</td>
</tr>
<tr>
<td>1.99594</td>
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<td>2.70459</td>
<td>2.70406</td>
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<tr>
<td>3.10795</td>
<td>3.10630</td>
</tr>
<tr>
<td>3.47820</td>
<td>3.48373</td>
</tr>
<tr>
<td>4.02964</td>
<td>4.02776</td>
</tr>
</tbody>
</table>

*Note: \( U = \frac{wG}{P} \) where \( P \) is the external force applied.*
Table 4.26  Highest Computed Displacement Amplitudes at Estimated Resonant Frequency Factors for $H/R = 0.30$, $H/L = 0.10$, $\nu = 0.30$, $\eta = 0.0$, and $n = 0$.

<table>
<thead>
<tr>
<th>Frequency Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>$U$</td>
</tr>
<tr>
<td>0.13951</td>
<td>190.80</td>
</tr>
<tr>
<td>0.19024</td>
<td>168.20</td>
</tr>
<tr>
<td>1.02090</td>
<td>2.46</td>
</tr>
<tr>
<td>1.85861</td>
<td>81.67</td>
</tr>
<tr>
<td>2.03234</td>
<td>14.90</td>
</tr>
<tr>
<td>3.00070</td>
<td>4.53</td>
</tr>
</tbody>
</table>

Table 4.27  Highest Computed Displacement Amplitudes at Estimated Resonant Frequency Factors for $H/R = 0.30$, $H/L = 0.10$, $\nu = 0.30$, $\eta = 0.0$, and $n = 1$.

<table>
<thead>
<tr>
<th>Frequency Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>$U$</td>
</tr>
<tr>
<td>0.09671</td>
<td>65.65</td>
</tr>
<tr>
<td>0.16487</td>
<td>94.88</td>
</tr>
<tr>
<td>0.26000</td>
<td>227.20</td>
</tr>
<tr>
<td>1.01773</td>
<td>0.29</td>
</tr>
<tr>
<td>1.04490</td>
<td>2.10</td>
</tr>
<tr>
<td>1.84220</td>
<td>95.21</td>
</tr>
</tbody>
</table>

*Note: $U = \frac{uG}{P_0H}$, $V = \frac{vG}{P_0H}$, $W = \frac{wG}{P_0H}$*
Table 4.28  Highest Computed Displacement Amplitudes at Estimated Resonant Frequency Factors for $H/R = 0.30$, $H/L = 0.10$ $\nu = 0.30$, $\eta = 0.0$, and $n = 2$.

<table>
<thead>
<tr>
<th>Frequency Factor $\Omega$</th>
<th>Highest Displacements $U$</th>
<th>$V$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.14427</td>
<td>113.00</td>
<td>101.10</td>
<td>27.00</td>
</tr>
<tr>
<td>0.22670</td>
<td>35.74</td>
<td>16.87</td>
<td>114.30</td>
</tr>
<tr>
<td>0.38220</td>
<td>69.05</td>
<td>75.19</td>
<td>44.25</td>
</tr>
<tr>
<td>1.02883</td>
<td>0.12</td>
<td>0.43</td>
<td>1.85</td>
</tr>
<tr>
<td>109540</td>
<td>7.63</td>
<td>33.98</td>
<td>13.65</td>
</tr>
<tr>
<td>1.81201</td>
<td>648.60</td>
<td>417.00</td>
<td>220.80</td>
</tr>
</tbody>
</table>

Table 4.29  Highest Computed Displacement Amplitudes at Estimated Resonant Frequency Factors for $H/R = 0.30$, $H/L = 0.10$ $\nu = 0.30$, $\eta = 0.0$, and $n = 3$.

<table>
<thead>
<tr>
<th>Frequency Factor $\Omega$</th>
<th>Highest Displacements $U$</th>
<th>$V$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16693</td>
<td>87.26</td>
<td>65.16</td>
<td>2.98</td>
</tr>
<tr>
<td>0.30914</td>
<td>11.82</td>
<td>9.53</td>
<td>64.97</td>
</tr>
<tr>
<td>0.51839</td>
<td>25.24</td>
<td>29.18</td>
<td>13.42</td>
</tr>
<tr>
<td>1.05260</td>
<td>0.09</td>
<td>0.42</td>
<td>1.13</td>
</tr>
<tr>
<td>1.16764</td>
<td>10.37</td>
<td>33.91</td>
<td>10.04</td>
</tr>
<tr>
<td>1.78191</td>
<td>59.48</td>
<td>48.98</td>
<td>17.23</td>
</tr>
</tbody>
</table>

*Note: $U = \frac{uG}{P_oH}$, $V = \frac{vG}{P_oH}$, $W = \frac{wG}{P_oH}$
Table 4.30  Highest Computed Displacement Amplitudes at Estimated
Resonant Frequency Factors for H/R = 0.30, H/L = 0.10
\( \nu = 0.30, \eta = 0.0, \) and \( n = 4. \)

<table>
<thead>
<tr>
<th>Frequency Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega )</td>
<td>( U )</td>
</tr>
<tr>
<td>0.18588</td>
<td>150.30</td>
</tr>
<tr>
<td>0.39830</td>
<td>4.69</td>
</tr>
<tr>
<td>0.65673</td>
<td>36.80</td>
</tr>
<tr>
<td>1.08275</td>
<td>0.11</td>
</tr>
<tr>
<td>1.25757</td>
<td>32.40</td>
</tr>
<tr>
<td>1.75735</td>
<td>39.60</td>
</tr>
</tbody>
</table>

Table 4.31  Variation of Displacements Amplitude with Respect to
Material Damping Factor (\( \eta \)) at the First Resonant
Frequency for H/R = 0.40, H/L = 0.10, \( \nu = 0.30, \) and
\( n = 0. \)

<table>
<thead>
<tr>
<th>Damping Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( U )</td>
</tr>
<tr>
<td>0.00</td>
<td>4.41</td>
</tr>
<tr>
<td>0.05</td>
<td>4.84</td>
</tr>
<tr>
<td>0.10</td>
<td>4.63</td>
</tr>
<tr>
<td>0.20</td>
<td>5.69</td>
</tr>
<tr>
<td>0.30</td>
<td>9.80</td>
</tr>
</tbody>
</table>

*Note: \( U = \frac{uG}{P_oH} \) \( \quad \) \( V = \frac{vG}{P_oH} \) \( \quad \) \( W = \frac{wG}{P_oH} \)
Table 4.32  Variation of Displacements Amplitude with Respect to Material Damping Factor (\(\eta\)) at the First Resonant Frequency for \(H/R = 0.40\), \(H/L = 0.10\), \(\nu = 0.30\), and \(n = 1\).

<table>
<thead>
<tr>
<th>Damping Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta)</td>
<td>(U)</td>
</tr>
<tr>
<td>0.00</td>
<td>243.00</td>
</tr>
<tr>
<td>0.05</td>
<td>35.90</td>
</tr>
<tr>
<td>0.10</td>
<td>19.28</td>
</tr>
<tr>
<td>0.20</td>
<td>10.15</td>
</tr>
<tr>
<td>0.30</td>
<td>6.98</td>
</tr>
</tbody>
</table>

Table 4.33  Variation of Displacements Amplitude with Respect to Material Damping Factor (\(\eta\)) at the First Resonant Frequency for \(H/R = 0.40\), \(H/L = 0.10\), \(\nu = 0.30\), and \(n = 2\).

<table>
<thead>
<tr>
<th>Damping Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta)</td>
<td>(U)</td>
</tr>
<tr>
<td>0.00</td>
<td>55.80</td>
</tr>
<tr>
<td>0.05</td>
<td>9.95</td>
</tr>
<tr>
<td>0.10</td>
<td>5.03</td>
</tr>
<tr>
<td>0.20</td>
<td>2.51</td>
</tr>
<tr>
<td>0.30</td>
<td>1.70</td>
</tr>
</tbody>
</table>

*Note:  \(U = \frac{uG}{P_0H}\),  \(V = \frac{vG}{P_0H}\),  \(W = \frac{wG}{P_0H}\)*
Table 4.34  Variation of Displacements Amplitude with Respect to Material Damping Factor (\(\eta\)) at the First Resonant Frequency for \(H/R = 0.40\), \(H/L = 0.10\), \(v = 0.30\), and \(n = 3\).

<table>
<thead>
<tr>
<th>Damping Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta)</td>
<td>(U)</td>
</tr>
<tr>
<td>0.00</td>
<td>161.10</td>
</tr>
<tr>
<td>0.05</td>
<td>8.42</td>
</tr>
<tr>
<td>0.10</td>
<td>8.05</td>
</tr>
<tr>
<td>0.20</td>
<td>6.64</td>
</tr>
<tr>
<td>0.30</td>
<td>5.54</td>
</tr>
</tbody>
</table>

Table 4.35  Variation of Displacements Amplitude with Respect to Material Damping Factor (\(\eta\)) at the First Resonant Frequency for \(H/R = 0.40\), \(H/L = 0.10\), \(v = 0.30\), and \(n = 4\).

<table>
<thead>
<tr>
<th>Damping Factor</th>
<th>Highest Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta)</td>
<td>(U)</td>
</tr>
<tr>
<td>0.00</td>
<td>159.70</td>
</tr>
<tr>
<td>0.05</td>
<td>16.20</td>
</tr>
<tr>
<td>0.10</td>
<td>8.20</td>
</tr>
<tr>
<td>0.20</td>
<td>4.20</td>
</tr>
<tr>
<td>0.30</td>
<td>2.88</td>
</tr>
</tbody>
</table>

*Note: \(U = \frac{uG}{P_oH}\), \(V = \frac{vG}{P_oH}\), \(W = \frac{wG}{P_oH}\).*
Table 4.36 Variation of Fundamental Resonant Frequency with Respect to Poisson's Ratio at Different H/L for Symmetric Radial Vibration (H/R = 1.0, n = 0).

<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>H/L</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.10</td>
</tr>
<tr>
<td>0.00</td>
<td>0.46054</td>
</tr>
<tr>
<td>0.10</td>
<td>0.14823</td>
</tr>
<tr>
<td>0.20</td>
<td>0.15314</td>
</tr>
<tr>
<td>0.25</td>
<td>0.15400</td>
</tr>
<tr>
<td>0.30</td>
<td>0.15299</td>
</tr>
<tr>
<td>0.35</td>
<td>0.15125</td>
</tr>
<tr>
<td>0.40</td>
<td>0.14807</td>
</tr>
<tr>
<td>0.45</td>
<td>0.14474</td>
</tr>
<tr>
<td>0.495</td>
<td>0.14173</td>
</tr>
</tbody>
</table>

Table 4.37 Variation of Fundamental Resonant Frequency with Respect to Poisson's Ratio at Different H/L for Rigid Body Motion (H/R = 1.0 and n = 1).

<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>H/L</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.10</td>
</tr>
<tr>
<td>0.00</td>
<td>0.03171</td>
</tr>
<tr>
<td>0.10</td>
<td>0.07610</td>
</tr>
<tr>
<td>0.20</td>
<td>0.08482</td>
</tr>
<tr>
<td>0.25</td>
<td>0.08800</td>
</tr>
<tr>
<td>0.30</td>
<td>0.09036</td>
</tr>
<tr>
<td>0.35</td>
<td>0.09750</td>
</tr>
<tr>
<td>0.40</td>
<td>0.09990</td>
</tr>
<tr>
<td>0.45</td>
<td>0.10067</td>
</tr>
<tr>
<td>0.495</td>
<td>0.11256</td>
</tr>
</tbody>
</table>
Table 4.38  Variation of Fundamental Resonant Frequency with Respect to Poisson's Ratio at Different H/L for the First Lobar Mode (H/R = 1.0, n = 2).

<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>H/L</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>v</td>
</tr>
<tr>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td></td>
</tr>
<tr>
<td>0.495</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.39  Variation of Fundamental Resonant Frequency with Respect to Poisson's Ratio at Different H/L for the Second Lobar Mode (H/R = 1.0, n = 3).

<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>H/L</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>v</td>
</tr>
<tr>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td></td>
</tr>
<tr>
<td>0.495</td>
<td></td>
</tr>
</tbody>
</table>
Table 4.40  Variation of Fundamental Resonant Frequency with Respect to Poisson's Ratio at Different H/L for the Third Lobar Mode (H/R = 1.0, n = 4).

<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>H/L 0.00</th>
<th>H/L 0.50</th>
<th>H/L 1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>v</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.91316</td>
<td>0.96072</td>
<td>1.16840</td>
</tr>
<tr>
<td>0.10</td>
<td>0.92901</td>
<td>0.97816</td>
<td>1.24290</td>
</tr>
<tr>
<td>0.20</td>
<td>0.94486</td>
<td>0.99401</td>
<td>1.30470</td>
</tr>
<tr>
<td>0.25</td>
<td>0.95120</td>
<td>1.00190</td>
<td>1.33327</td>
</tr>
<tr>
<td>0.30</td>
<td>0.95912</td>
<td>1.00984</td>
<td>1.36170</td>
</tr>
<tr>
<td>0.35</td>
<td>0.96457</td>
<td>1.02410</td>
<td>1.38559</td>
</tr>
<tr>
<td>0.40</td>
<td>0.97340</td>
<td>1.08120</td>
<td>1.40778</td>
</tr>
<tr>
<td>0.45</td>
<td>0.97974</td>
<td>1.13986</td>
<td>1.42205</td>
</tr>
<tr>
<td>0.495</td>
<td>0.98450</td>
<td>1.17316</td>
<td>1.43010</td>
</tr>
</tbody>
</table>
Fig. 4.1 First six resonant frequency factors for $H/R = 0.30$ and $n = 0$. 
Fig. 4.2 First six resonant frequency factors for $H/R = 0.30$ and $n = 1$
Fig. 4.3 First six resonant frequency factors for $H/R = 0.30$ and $n = 2$
Fig. 4.4 First six resonant frequency factors for H/R = 0.30 and n = 3
Fig. 4.5 First six resonant frequency factors for $H/R = 0.30$ and $n = 4$
Fig. 4.6 First six resonant frequency factors for H/R = 0.40 and n = 0
Fig. 4.7 First six resonant frequency factors for H/R = 0.40 and n = 1
Fig. 4.8 First six resonant frequency factors for \( H/R = 0.40 \) and \( n = 2 \)
Fig. 4.9 First six resonant frequency factors for $H/R = 0.40$ and $n = 3$
Fig. 4.10  First six resonant frequency factors for \( H/R = 0.40 \) and \( n = 4 \)
Fig. 4.11 First six resonant frequency factors for $H/R = 0.50$ and $n = 0$
Fig. 4.12 First six resonant frequency factors for $H/R = 0.50$ and $n = 1$
Fig. 4.13 First six resonant frequency factors for H/R = 0.50 and n = 2
Fig. 4.14 First six resonant frequency factors for $H/R = 0.50$ and $n = 3$
Fig. 4.15 First six resonant frequency factors for $H/R = 0.50$ and $n = 4$
Fig. 4.16 First six resonant frequency factors for $H/R = 1.00$ and $n = 0$
Fig. 4.17  First six resonant frequency factors for $H/R = 1.00$ and $n = 1$
Fig. 4.18 First six resonant frequency factors for $H/R = 1.00$ and $n = 2$
Fig. 4.19 First six resonant frequency factors for H/R = 1.00 and n = 3
Fig. 4.20  First six resonant frequency factors for $H/R = 1.00$ and $n = 4$
Fig. 4.21 First six resonant frequency factors for $H/R = 1.90$ and $n = 0$
Fig. 4.22 First six resonant frequency factors for $H/R = 1.90$ and $n = 1$
Fig. 4.23  First six resonant frequency factors for $H/R = 1.90$ and $n = 2$
Fig. 4.24 First six resonant frequency factors for $H/R = 1.90$ and $n = 3$
Fig. 4.25 First six resonant frequency factors for $H/R = 1.90$ and $n = 4$
Fig. 4.26 Variation of amplitude with respect to damping ratio for $H/R = 0.40$, $H/L = 0.10$ and $n = 0$
Fig. 4.27 Variation of amplitude with respect to damping ratio for $H/R = 0.40$, $H/L = 0.10$ and $n = 1$
Fig. 4.28 Variation of amplitude with respect to damping ratio for $H/R = 0.40$, $H/L = 0.10$ and $n = 2$
Fig. 4.29 Variation of amplitude with respect to damping ratio for H/R = 0.40, H/L = 0.10 and n = 3
Fig. 4.30 Variation of amplitude with respect to damping ratio for $H/R = 0.40$, $H/L = 0.10$ and $n = 4$
Fig. 4.31 Variation of fundamental resonant frequency factor with respect to Poisson's ratio for $H/R = 1.00$ and $n = 0$. 
Fig. 4.32 Variation of fundamental resonant frequency factor with respect to Poisson's ratio for \( H/R = 1.00 \) and \( n = 1 \)
Fig. 4.33 Variation of fundamental resonant frequency factor with respect to Poisson's ratio for $H/R = 1.00$ and $n = 2$. 
Fig. 4.34 Variation of fundamental resonant frequency factor with respect to Poisson's ratio for $H/R = 1.00$ and $n = 3$
In this chapter, the conclusions and recommendations for further research are presented.

5.1 Conclusions

The forced vibrations of isotropic, elastic hollow cylinders have been studied. In particular, by considering the propagation of the types of elastic waves, the dynamic response for hollow, elastic cylinders has been provided. The dynamic response has been obtained using the three-dimensional theory of elasticity. As a result, the results obtained in this manner are valid for both thin and thick cylinders, but these results are valid to the limits for solid cylinders. By assuming harmonic excitations in infinitely long hollow cylinders, the resonant frequencies for a wide range of internal and external modes have been obtained. The resonant frequencies are presented in the form of frequency charts.

The resonant frequencies obtained in this fashion have been confirmed by the natural frequencies obtained by Osawa in 1959. This result is in conjunction with Almenahal and Manesson in 1988. These results, which have been subject to slight correction, are the fundamental

Fig. 4.35 Variation of fundamental resonant frequency factor with respect to Poisson's ratio for H/R = 1.00 and n = 4
In this chapter, the conclusions and recommendations for further research are presented.

5.1 Conclusions

The forced vibrations of isotropic, elastic hollow cylinders has been studied. In particular, by considering the propagation of two types of elastic waves, the dynamic response for hollow, elastic cylinders has been provided. The dynamic response has been obtained using the three-dimensional theory of elasticity. As a result, the results obtained in this investigation are valid for both thin and thick cylinders. In fact, these results are valid in the limit for solid cylinders. By assuming harmonic excitations in infinitely long hollow cylinders, the resonant frequencies for a wide range of thicknesses and lobar modes have been provided. These resonant frequencies are presented in several tables and frequency charts. These tables and charts can be of great assistance for future engineering design of cylindrical structures.

The resonant frequencies obtained in this research have been compared to the natural frequencies obtained by Gazis in 1959 (published in conjunction with Armenakas and Herrmann in 1968). These results have been shown to agree except for the fundamental
frequency when H/L approaches unity. These differences have been attributed to the computations of the Bessel functions values for both investigations.

The variation of the fundamental frequency with respect to Poisson's ratio have been found for different values of circumferential waves. The effect of varying Poisson's ratio over the range zero to one half have been presented. The values of the fundamental frequency have been shown to increase with increasing Poisson's ratio.

The variation of displacements amplitudes with respect to material damping ratio have been considered. The effect of varying material damping ratio between $\eta = 0$ and $\eta = 0.30$ for different values of circumferential waves have been studied. As expected, the amplitudes of displacements have been shown to decrease linearly with increasing material damping.

Finally, the coupling effect for the first three modes has been discussed. This discussion was done over a wide range of H/R and H/L ratios, as well as over five different numbers of circumferential waves.

5.2 Recommendations for Further Research

At the end of this investigation, the author would like to present some considerations for further studies in the vibrations of hollow cylinders.

First of all, it should be mentioned that the solution to the vibrations of hollow, elastic cylinders presented in Chapter Three is
in a general form (see equations 3.40). As mentioned earlier, assuming infinitely long cylinders, only six boundary conditions need to be satisfied. The vibrations of infinitely long cylinders have been studied in this investigation. However, the method of analysis presented herein can be extended to study the vibrations of finite size cylinders, providing the appropriate boundary conditions are satisfied.

Experimental studies should be undertaken to determine the natural frequencies of hollow cylinders. This would allow a comparison between the experimental and theoretical results. Most of the experimental data available nowadays is limited to very thin shells. Hence, experimental determination of resonant frequencies for thicker cylinders is vitally needed.

Another area for future research is the damping treatment on hollow cylinders for all possible modes. The theory of viscoelastic damping have been widely in the vibration attenuation of the radial and torsional modes. However, the damping of axial modes has not been successfully undertaken.

Based on the theoretical technique presented herein, an on-line early warning system for identification of fatigue cracks in cylindrical structures, such as piping and containment vessels in nuclear power plants can be developed.
REFERENCES


The computer program used in most of the computations in this investigation is listed in the pages to follow. The explanations of the input and output parameters are depicted as comment statements throughout the computer coding.

As mentioned in Chapter Four, this program was used to compute the resonant frequency factors (Tables 4.1 to 4.25). The results listed in tables 4.26 to 4.40 were also obtained by using a modified version of this program. The necessary modifications were outlined in Chapter Four.
** INITIALIZE: **
** ROW = Mass density of the material **
** G = Shear modulus of the material **
** NU = Poisson's ratio of the material **
** H = Thickness of the cylinder **
** HDR = Thickness to mean radius ratio **
** HDL = Thickness to length ratio **
** N = Number of circumferential waves **
** OME = Exciting frequency in Hz **
** DOME= Increment of exciting frequency (Hz) **

ROW=0.000725
NU=0.30
G=(11538461.0,0.0)
H=1.00
HDR=0.50
N=0
HDL=1.0E-10
OME=10.0
DOME=5.0

** COMPUTE: **
** LAM = Lame's constant **
** V1 = Velocity of dilatational waves **
** V2 = Velocity of distortional waves **
** GAMA = Wave number in axial direction **
** R = Mean radius **
** RA = Inside radius **
** RB = Outside radius **

LAM=2.0*NU/(1.0-2.*NU)*G
V1=CSQRT((LAM+2.*G)/ROW)
V2=CSQRT(G/ROW)
GAMA=PIE/(H/HDL)
R=H/HDR
RA=R-H/2.
RB=R+H/2.

** Specify boundaries stresses **

**
DO 10 I=1,6
SIGMA(I)=(0.,0.)
SIGMA(I)=(1000.,0.)

WRITE(1,900)NU
WRITE(1,910)RA,H,R
WRITE(1,920)HDL,N,HDL
DO 20 I=1,1000

OME=(OME+DOME)*2.0*PIE
XI=CSQR((OME/V2)**2-GAMA**2)
PHI=CSQR((OME/V1)**2-GAMA**2)

CALL SMAT(N,OME,RA,RA,SI,S)
CALL TMAT(N,RA,OME,T)
CALL MET204(S,6,SIGMA,A,SI,ER)
CALL MET113(T,A,9,6,U)

OME=OME/(2.*PIE)
DF=2.*H*OME/V2
DO 15 J=1,9
UABS(J)=CABS(U(J))
WRITE(1,930)DF,(UABS(JR),JR=1,9)
CONTINUE

FORMAT(T10,POISSONS RATIO=',F4.2)
FORMAT(T10,'H/R=',F5.3,5X,'N=',I2,5X,'H/L=',E10.3)
FORMAT(F7.5,E12.4)
**SMAT**

**THIS SUBROUTINE GENERATES MATRIX S FOR**

**GIVEN VALUES OF:**

**N = NUMBER OF CIRCUMFERENCE WAVES**

**OME = EXCITING FREQUENCY**

**RA = INSIDE RADIUS**

**RB = OUTSIDE RADIUS**

**SUBROUTINE SMAT(N,OME,RA,RB,S)**

COMMON V1,V2,XI,PHI,GAMA,LAM,G

COMPLEX V1,V2,WN,DWN,ZN,DZN,D2ZN

COMPLEX S(6,6),LAM,G,XI,PHI

XN=FLOAT(N)

XA=CABS(XI)

PA=CABS(PHI)

DO 5 I=1,6

DO 5 J=1,6

S(I,J)=(0.,0.)

R=RA

DO 10 I=1,2

CALL BESGEN(1,N,R,WN,DWN,ZN,DZN,D2ZN,OME)

S(I,1)=-LAM*(PA**2+GAMA**2)*ZN+2.*G*PA**2*D2ZN

S(I,4)=-LAM*(PA**2+GAMA**2)*WN+2.*G*PA**2*D2WN

S(I+2,1)=-2.*G*XI*(DZN*PA/R-ZN/R**2)

S(I+2,4)=-2.*G*XI*(DWN*PA/R-WN/R**2)

S(I+4,1)=-2.*G*GAMA*PA*DZN

S(I+4,4)=-2.*G*GAMA*PA*DWN

R=RB

R=RA

DO 20 I=1,2

CALL BESGEN(2,N,R,WN,DWN,ZN,DWN,D2WN,OME)

S(I,2)=2.*G*XI*(XA*DZN/R-ZN/R**2)

S(I,5)=2.*G*XI*(XA*DWN/R-WN/R**2)

S(I+2,2)=-G*(XA**2*D2ZN-XA*DZN/R+XI*WN*ZN/R**2)

S(I+2,5)=-G*(XA**2*D2WN-XA*DWN/R+XI*WN*WN/R**2)

S(I+4,2)=-G*GAMA*XA*WN/R

S(I+4,5)=-G*GAMA*WN/XN/R

R=RB

R=RA

XN=XN+1.0

DO 30 I=1,2

CALL BESGEN(2,N+1,R,WN,DWN,D2WN,ZN,DZN,D2ZN,OME)

S(I,3)=2.*G*GAMA*XA*DZN

S(I,6)=2.*G*GAMA*XA*DWN

S(I+2,3)=G*GAMA*(XA*DZN-XN*ZN/R)

S(I+2,6)=G*GAMA*(XA*DWN-XN*WN/R)

S(I+4,3)=G*(WN*ZN/R**2-GAMA**2*ZN-XN*XA*DZN/R-XA**2*D2ZN)

S(I+4,6)=G*(WN*WN/R**2-GAMA**2*WN-XN*XA*DWN/R-XA**2*D2WN)

R=RB
** THIS SUBROUTINE GENERATES MATRIX T GIVEN: **
** N = NUMBER OF CIRCUMFERENTIAL WAVES **
** R = ANY RADIUS IN THE CYLINDER **
** OME = EXCITING FREQUENCY **

**

SUBROUTINE TMAT(N,R,OME,T)
COMMON V1,V2,XI,PHI,LAM,G
COMPLEX V1,V2,WN,DWN,ZN,DZN,D2N
COMPLEX T(9,6),XI,PHI,LAM,G
XN=FLOAT(N)
XA=CABS(XI)
PA=CABS(PHI)
DO 5 I=1,9
DO 5 J=1,6
T(1,J)=(0.,0.)
CALL BESGEN(1,N,R,WN,DWN,D2WN,ZN,DZN,D2ZN,OME)
T(1,1)=LAM*(PA**2+GAMA**2)*ZN+2.*G*PA*D2ZN
T(1,4)=LAM*(PA**2+GAMA**2)*WN+2.*G*PA*D2WN
T(2,1)=(LAM*(PA**2+GAMA**2)+2.*G*XN**2/R**2)*ZN+2.*G*PA*DZN/R
T(2,4)=T(2,1)
T(2,2)=(LAM*(PA**2+GAMA**2)+2.*G*XN**2/R**2)*WN-2.*G*PA*DWN/R
T(2,3)=T(2,4)
T(3,1)=(-LAM*(PA**2+GAMA**2)-2.*G*GAMA**2)*ZN
T(3,4)=(-LAM*(PA**2+GAMA**2)-2.*G*GAMA**2)*WN
T(4,1)=2.*G*XN*(DZN*PA/R-ZN/R**2)
T(4,4)=2.*G*XN*(DWN*PA/R-WN/R**2)
T(5,1)=2.*G*GAMA*PA*DZN
T(5,4)=2.*G*GAMA*PA*DWN
T(6,1)=2.*G*XN*GAMA*ZN/R
T(6,4)=2.*G*XN*GAMA*WN/R
T(7,1)=PA*DZN
T(7,4)=PA*DWN
T(8,1)=-XN*ZM/R
T(8,4)=-XN*WN/R
T(9,1)=-GAMA*ZN
T(9,4)=-GAMA*WN
CALL BESGEN(2,N,R,WN,DWN,D2WN,ZN,DZN,D2ZN,OME)
T(1,2)=2.*G*XN*(XA*DZN/R-ZN/R**2)
T(1,5)=2.*G*XN*(XA*DWN/R-WN/R**2)
T(2,2)=T(1,2)
T(2,5)=T(1,5)
T(4,2)=G*(XA**2*D2ZN-XA*DZN/R+XN*XN*ZN/R**2)
T(4,5)=G*(XA**2*D2WN-XA*DWN/R+XN*XN*WN/R**2)
T(5,2)=G*XN*GAMA*ZN/R
T(5,5)=G*XN*GAMA*WN/R
T(6,2)=G*GAMA*XA*DZN
T(6,5)=G*GAMA*XA*DWN
\[ T(7,2) = XN \cdot ZN / R \]
\[ T(7,5) = XN \cdot WN / R \]
\[ T(8,2) = -XA \cdot DZN \]
\[ T(8,5) = -XA \cdot DWN \]

\[ XN = XN + 1.0 \]

CALL BESGEN(2, N+1, R, WN, DWN, D2WN, ZN, DZN, D2ZN, OME)

\[ T(1,3) = 2.0 \cdot G \cdot GAMA \cdot XA \cdot DZN \]
\[ T(1,6) = 2.0 \cdot G \cdot GAMA \cdot XA \cdot DWN \]
\[ T(2,3) = 2.0 \cdot G \cdot GAMA \cdot XN \cdot ZN / R \]
\[ T(2,6) = 2.0 \cdot G \cdot GAMA \cdot XN \cdot WN / R \]
\[ T(3,3) = -2.0 \cdot G \cdot GAMA \cdot (XN \cdot ZN / R + XA \cdot DZN) \]
\[ T(3,6) = -2.0 \cdot G \cdot GAMA \cdot (XN \cdot WN / R + XA \cdot DWN) \]
\[ T(4,3) = G \cdot GAMA \cdot (XA \cdot DZN - XN \cdot ZN / R) \]
\[ T(4,6) = G \cdot GAMA \cdot (XA \cdot DWN - XN \cdot WN / R) \]
\[ T(5,3) = G \cdot (XN \cdot ZN / R + XN \cdot XA \cdot DZN / R - XA \cdot D2ZN) \]
\[ T(5,6) = G \cdot (XN \cdot WN / R + XN \cdot XA \cdot DWN / R - XA \cdot D2WN) \]

\[ XN = XN - 1.0 \]
\[ T(6,3) = G \cdot (XN \cdot XA \cdot DZN / R + (XN \cdot ZN / R + XN \cdot XA \cdot DZN / R) - GAMA \cdot XN \cdot ZN) \]
\[ T(6,6) = G \cdot (XN \cdot XA \cdot DWN / R + (XN \cdot WN / R + XN \cdot XA \cdot DWN / R) - GAMA \cdot XN \cdot WN) \]
\[ T(7,3) = GAMA \cdot ZN \]
\[ T(7,6) = GAMA \cdot WN \]
\[ T(8,3) = T(7,3) \]
\[ T(8,6) = T(7,6) \]
\[ T(9,3) = (XA \cdot DZN + ZN \cdot (XN + 1.0) / R) \]
\[ T(9,6) = (XA \cdot DWN + WN \cdot (XN + 1.0) / R) \]

RETURN

END

*************************************************************************
**                         BESGEN                         **
** This subroutine computes the appropriate Bessel **
** functions and their derivatives for given: **
** K = Flag indicating argument of Bessel function **
** (see table 3.1) **
** N = Number of circumferential waves **
** R = R-coordinate of a point in the cylinder **
** OME = Exciting frequency **
**
*************************************************************************

SUBROUTINE BESGEN(K, N, R, WN, DWN, D2WN, ZN, DZN, D2ZN, OME)
COMMON V1, V2, XI, PHI, GAMA, LAM, G
COMPLEX V1, V2, Z, AR, G, LAM, XI, PHI
COMPLEX WN, DWN, D2WN, ZN, DZN, D2ZN
XN=FLOAT(N)
IF(K.EQ.2)GO TO 10
AR=PHI
GO TO 20
AR=XI
10 X=R*REAL(AR)
Y=R*AIMAG(AR)
Z=CMPLX(X, Y)
X=CABS(Z)
Y=0.0
T1 = CABS(V1*GAMA)
T2 = CABS(V2*GAMA)
IF(OME .GT. T1) GO TO 100
IF(OME .GT. T2 .AND. OME .LT. T1) GO TO 200
CALL KAI(WN,DWN,D2WN,ZN,DZN,D2ZN,N,X,Y)
RETURN
CALL JAY(ZN,DZN,D2ZN,WN,DWN,D2WN,N,X,Y)
RETURN
IF(K.EQ.2) GO TO 100
GO TO 60
END

** ******************* M E T 1 0 9 ******************* **
** MULTIPLICATION OF A COMPLEX SQUARE MATRIX **
** BY A COMPLEX VECTOR **
** ******************* M E T 1 0 9 ******************* **

SUBROUTINE MET109 (A,B,N,C)
COMPLEX A(N,N),B(N),C(N)
DO 10 I=1,N
C(I) = (0.,0.)
DO 10 J=1,N
10 C(I) = C(I) + A(I,J)*B(J)
RETURN
END

** ******************* M E T 1 1 3 ******************* **
** MULTIPLICATION OF A RECTANGULAR MATRIX **
** BY A VECTOR (COMPLEX) **
** ******************* M E T 1 1 3 ******************* **

SUBROUTINE MET113 (A,V,N,M,C)
COMPLEX A(N,M),V(M),C(N)
DO 10 I=1,N
C(I) = (0.,0.)
DO 10 J=1,M
10 C(I) = C(I) + A(I,J)*V(J)
CONTINUE
RETURN
END

** ******************* M E T 1 2 2 ******************* **
** INVERSION OF A COMPLEX MATRIX **
** (PIVOTING) **
** ******************* M E T 1 2 2 ******************* **

SUBROUTINE MET122 (A,B,N,ER)
COMPLEX A(N,N),B(N,N)
COMPLEX CC,D,S,F
DO 3 I=1,N
DO 3 J=1,N
IF(I-J)1,2,1
1 B(I,J)=(0.,0.)
GO TO 3
2 B(I,J)=(1.,0.)
CONTINUE
DO 11 J=1,N
DO 6 I=J,N
IF(CABS(A(J,J))-CABS(A(I,J)))4,6,6
DO 5 K=1,N
CC=A(I,K)
A(I,K)=A(J,K)
A(J,K)=CC
D=B(I,K)
B(I,K)=B(J,K)
B(J,K)=D
CONTINUE
6 CONTINUE
IF(CABS(A(J,J))-ER)14,14,7
DO 10 I=1,N
IF(I-J)8,10,8
S=A(I,J)/A(J,J)
DO 9 L=1,N
A(I,L)=A(J,L)*S-A(I,L)
B(I,L)=B(J,L)*S-B(I,L)
9 CONTINUE
10 CONTINUE
11 CONTINUE
DO 13 I=1,N
F=A(I,I)
A(I,I)=(1.,0.)
DO 12 J=1,N
B(I,J)=B(I,J)/F
12 CONTINUE
13 CONTINUE
GO TO 16
14 WRITE(6,15)
15 FORMAT(///,10X,'MATRIX IS SINGULAR')
STOP
16 RETURN
END

**********************************************************************************
**                         M E T  2 0 4                                    **
** SOLUTION FOR A COMPLEX SYSTEM OF EQUATIONS                    **
** (NxN). "PIVOTING"                          **
**********************************************************************************

SUBROUTINE MET204(A,N,B,X,AL,ER)
COMPLEX A(N,N),AL(N,N),B(N),X(N)
CALL MET122 (A,AL,N,ER)
CALL MET109 (AL,B,N,X)
RETURN
SUBROUTINE JAY(JN, DJN, D2JN, YN, DYN, D2YN, N, X, Y)
DIMENSION BJRE(100), BJIM(100), YRE(100), YIM(100)
COMPLEX JN, DJN, D2JN, YN, DYN, D2YN, Z
XN=FLOAT(N)
Z=Cmplx(X, Y)
IF(N. NE. 0) GO TO 1
CALL COMBES(X, Y, O.. O., 1, BJRE, BJIM, YRE, YIM)
JN=Cmplx(BJRE(1) , BJIM(1))
YN=Cmplx(YRE(1) , YIM(1))
DJN=Cmplx(BJRE(2), BJIM(2))
DYN=Cmplx(YRE(2), YIM(2))
D2JN=-JN/DJN/Z+Cmplx(BJRE(N), BJIM(N))
D2YN=-YN/DYN/Z+Cmplx(YRE(N), YIM(N))
RETURN
1 CALL COMBES(X, Y, O.. O., N, BJRE, BJIM, YRE, YIM)
JN=Cmplx(BJRE(N+1), BJIM(N+1))
YN=Cmplx(YRE(N+1), YIM(N+1))
DJN=-XN*JN/Z+Cmplx(BJRE(N), BJIM(N))
DYN=-XN*YN/Z+Cmplx(YRE(N), YIM(N))
D2JN=(-Z**2*XN+XN**2)/Z**2*JN-Cmplx(BJRE(N), BJIM(N))/Z
D2YN=(-Z**2*XN+XN**2)/Z**2*YN-Cmplx(YRE(N), YIM(N))/Z
RETURN
END
SUBROUTINE COMBES(X, Y, ALPHA, BETA, N, BJRE, BJIM, YRE, YIM)
DIMENSION BJRE(100), BJIM(100), YRE(100), YIM(100)
CALL START1(X, Y, N, K, R)
CALL JRECUR(X, Y, ALPHA, BETA, K, R, BJRE, BJIM)
CALL JSUM(ALPHA, K, BJRE, BJIM, SUMRA, SUMIA)
CALL FACTOR(X, Y, ALPHA, BETA, Q, R)
CALL JNORM(K, Q, R, SUMRA, SUMIA, BJRE, BJIM)
CALL YSUM(X, Y, ALPHA, BETA, K, BJRE, BJIM, ASUMR, ASUMI)
CALL YGNU(X, Y, ALPHA, BETA, Q, R, ASUMR, ASUMI, BJRE, BJIM, YRE, YIM)
CALL WRONSK(X, Y, BJRE, BJIM, YRE, YIM)
IF(N. NE. 0) GO TO 12
RETURN
12 CALL YRECUR(X, Y, N, BJRE, BJIM, YRE, YIM)
RETURN
SUBROUTINE WRONSK(X,Y,BJRE,BJIM,YRE,YIM)
DIMENSION BJRE(100),BJIM(100),YRE(50),YIM(50)
SSQ=X*X+Y*Y
TPI=2.0/3.141592654
AZRE=TPI*X/SSQ
AZIM=-TPI*Y/SSQ
ZRE=BJRE(2)*YRE(1)-BJIM(2)*YIM(1)
ZIM=BJIM(2)*YRE(1)+BJRE(2)*YIM(1)
BZRE=ZRE-AZRE
BZIM=ZIM-AZIM
BJSQ=BJRE(1)*BJRE(1)+BJIM(1)*BJIM(1)
CZRE=BJRE(1)/BJSQ
CZIM=(-BJIM(1))/BJSQ
YRE(2)=BZRE*CZRE-BZIM*CZIM
YIM(2)=BZIM*CZRE+BZRE*CZIM
RETURN
END

SUBROUTINE NEGN(X,Y,ALPHA,BETA,N,BJRE,BJIM,YRE,YIM)
DIMENSION BJRE(100),BJIM(100),YRE(50),YIM(50)
L=IABS(N)+1
SSQ=X*X+Y*Y
TX=2.0*X
TY=2.0*Y
RALPHA=ALPHA
A=(TX*RALPHA+TY*BETA)/SSQ
B=(-TY*RALPHA+TX*BETA)/SSQ
BJRE(2)=A*BJRE(1)-B*BJIM(1)-BJRE(2)
BJIM(2)=B*BJRE(1)+A*BJIM(1)-BJIM(2)
YRE(2)=A*YRE(1)-B*YIM(1)-YRE(2)
YIM(2)=B*YRE(1)+A*YIM(1)-YIM(2)
DO 1 I=3,L
RALPHA=RALPHA-1.0
A=(TX*RALPHA+TY*BETA)/SSQ
B=(-TY*RALPHA+TX*BETA)/SSQ
BJRE(I)=A*BJRE(I-1)-B*BJIM(I-1)-BJRE(I-2)
BJIM(I)=B*BJRE(I-1)+A*BJIM(I-1)-BJIM(I-2)
YRE(I)=A*YRE(I-1)-B*YIM(I-1)-YRE(I-2)
YIM(I)=B*YRE(I-1)+A*YIM(I-1)-YIM(I-2)
1 RETURN
END

SUBROUTINE JRECUR(X,Y,ALPHA,BETA,K,R,BJRE,BJIM)
DIMENSION BJRE(100),BJIM(100)
RALPHA=R+ALPHA
SSQ=X*X+Y*Y
BJRE(K+2)=0.
BJIM(K+2)=0.
BJRE(K+1)=1.0E-37
BJIM(K+1)=0.0
DO 4 I=1,K
L1=K+1-I
RALPHA=RALPHA-1.0
A=((2.0*X*RALPHA)+(2.0*BETA*Y))/SSQ
4 CONTINUE
B = ((-2.0*Y*RALPHA)+(2.0*BETA*X))/SSQ
BJRE(L1) = (A*BJRE(L1+1))-(B*BJIM(L1+1))-BJRE(L1+2)
BJIM(L1) = (B*BJRE(L1+1))+(A*BJIM(L1+1))-BJIM(L1+2)
RETURN
END

SUBROUTINE JSUM(ALPHA, BETA, K, BJRE, BJIM, SUMRA, SUMIA)
DIMENSION BJRE(100), BJIM(100)
SUMRA = (BJRE(3)*(ALPHA+2.0))-(BJIM(3)*BETA)
SUMIA = (BETA*BJRE(3))+(ALPHA+2.0)*BJIM(3))
GRE = 1.0
GIM = 0.
S = 1.0
DO 6 I = 5, K, 2
S = S+1.0
GRE = ((GRE*(ALPHA+S-1.0))-(BETA*GIM))/S
GIM = ((GIM*(ALPHA+S-1.0))+(BETA*GRE))/S
GRE = GRE
ALPTS = ALPHA+2.0*S
GJR = GRE*BJRE(I)
GJI = GIM*BJIM(I)
GJRI = GRE*BJIM(I)
GJIR = GIM*BJRE(I)
SUMRB = ALPTS*(GJR-GJI)-BETA*(GJIR+GJRI)+SUMRA
SUMIB = ALPTS*(GJIR+GJRI)-BETA*(GJI-GJR)+SUMIA
IF(ABS((SUMRB/SUMRA)-1.0)-0.00000005)21, 21, 10
IF(ABS(SUMIA)-0.00000005)11, 11, 20
20 IF(ABS((SUMIB/SUMIA)-1.0)-0.00000005)11, 11, 10
10 SUMRA = SUMRB
6 SUMIA = SUMIB
11 RETURN
END

SUBROUTINE JNORM(K, Q, R, SUMRA, SUMIA, BJRE, BJIM)
DIMENSION BJRE(100), BJIM(100)
S = ((SUMRA+BJRE(1))*Q)-((SUMIA+BJIM(1))*R)
T = ((SUMIA+BJIM(1))*Q)+((SUMRA+BJRE(1))*R)
IF(ABS(S)-ABS(T))100, 101, 101
101 TS = T/S
TSSQ = S*(1.0+(TS*TS))
12 DO 13 I = 1, K
BJREN = (BJRE(I)+BJIM(I)*TS)/TSSQ
BJIM(I) = (BJIM(I)-BJRE(I)*TS)/TSSQ
13 BJRE(I) = BJREN
GO TO 14
100 ST = S/T
STSQ = T*((ST*ST)+1.0)
102 DO 103 I = 1, K
BJREN = (BJRE(I)*ST+BJIM(I))/STSQ
BJIM(I) = (BJIM(I)*ST-BJRE(I))/STSQ
103 BJRE(I) = BJREN
14 RETURN
END

SUBROUTINE YSUM(X, Y, ALPHA, BETA, K, BJRE, BJIM, ASUMR, ASUMI)
DIMENSION BJRE(100), BJIM(100)
A1=\text{ALPHA}-1.0
A2=A1-1.0
A3=A1+\text{ALPHA}
A4=\text{BETA} \times \text{BETA}
A5=2.0 \times A4
\text{ABSQ}=(-A1) \times (-A1) + A4
\text{GAMRE}=((2.0+\text{ALPHA}) \times (-A1)-A4)/\text{ABSQ}
\text{GAMIM}=(\text{BETA} \times 3.0)/\text{ABSQ}
\text{ASUMR}=\text{GAMRE} \times \text{BJRE}(3) - \text{GAMIM} \times \text{BJIM}(3)
\text{ASUMI}=\text{GAMIM} \times \text{BJRE}(3) + \text{GAMRE} \times \text{BJIM}(3)

T=1.0
\text{DO } 500 \text{ I}=5,K,2
\text{DO } 500 \text{ I}=5,K,2
\text{T}=\text{T}+1.0
\text{T}=\text{T}+1.0
\text{B1}=2.0 \times \text{T}
\text{B1}=2.0 \times \text{T}
\text{F1}=\text{B1}+\text{ALPHA}
\text{F1}=\text{B1}+\text{ALPHA}
\text{F2}=A3+\text{T}
\text{F2}=A3+\text{T}
\text{F3}=A1+\text{T}
\text{F3}=A1+\text{T}
\text{F5}=\text{T}-\text{ALPHA}
\text{F5}=\text{T}-\text{ALPHA}
\text{F6}=A2+B1
\text{F6}=A2+B1
\text{G1}=\text{F1} \times \text{F2}-A5
\text{G1}=\text{F1} \times \text{F2}-A5
\text{G2}=(\text{F2}+2.0 \times \text{F1}) \times \text{BETA}
\text{G2}=(\text{F2}+2.0 \times \text{F1}) \times \text{BETA}
\text{H1}=\text{G1} \times \text{F3} - G2 \times \text{BETA}
\text{H1}=\text{G1} \times \text{F3} - G2 \times \text{BETA}
\text{H2}=G2 \times \text{F3} + \text{G1} \times \text{BETA}
\text{H2}=G2 \times \text{F3} + \text{G1} \times \text{BETA}
\text{P1}=\text{F5} \times \text{F6} + A4
\text{P1}=\text{F5} \times \text{F6} + A4
\text{P2}=\text{F5} \times \text{F6} \times \text{BETA}
\text{P2}=\text{F5} \times \text{F6} \times \text{BETA}
\text{P3}=\text{P1} \times \text{P1} + \text{F2} \times \text{F2}
\text{P3}=\text{P1} \times \text{P1} + \text{F2} \times \text{F2}
\text{CRE}=((\text{H1} \times \text{P1}+\text{H2} \times \text{P2})/\text{F3})/\text{T}
\text{CRE}=((\text{H1} \times \text{P1}+\text{H2} \times \text{P2})/\text{F3})/\text{T}
\text{CIM}=((\text{H2} \times \text{P1}-\text{H1} \times \text{P2})/\text{F3})/\text{T}
\text{CIM}=((\text{H2} \times \text{P1}-\text{H1} \times \text{P2})/\text{F3})/\text{T}
\text{TEMP}=-(\text{CRE} \times \text{GAMRE}-\text{CIM} \times \text{GAMIM})
\text{TEMP}=-(\text{CRE} \times \text{GAMRE}-\text{CIM} \times \text{GAMIM})
\text{GAMRE}=\text{TEMP}
\text{GAMRE}=\text{TEMP}
\text{BSUMR}=\text{GAMRE} \times \text{BJRE}(I)-\text{GAMIM} \times \text{BJIM}(I)+\text{ASUMR}
\text{BSUMR}=\text{GAMRE} \times \text{BJRE}(I)-\text{GAMIM} \times \text{BJIM}(I)+\text{ASUMR}
\text{BSUMI}=\text{GAMIM} \times \text{BJRE}(I)+\text{GAMRE} \times \text{BJIM}(I)+\text{ASUMI}
\text{BSUMI}=\text{GAMIM} \times \text{BJRE}(I)+\text{GAMRE} \times \text{BJIM}(I)+\text{ASUMI}
\text{IF}(\text{ABS}\left((\text{BSUMR}/\text{ASUMR})-1.0\right)-.00000005)521,521,510
\text{IF}(\text{ABS}(\text{ASUMI})-0.000000005)511,511,520
\text{IF}(\text{ABS}\left((\text{BSUMI}/\text{ASUMI})-1.0\right)-.00000005)511,511,510
\text{IF}(\text{ALPHA})1,2,1
\text{IF}(\text{BETA})1,3,1
\text{IF}(\text{ALPHA})1,2,1
\text{IF}(\text{BETA})1,3,1
\text{CALL YZERO}(X,Y,\text{ALPREF},\text{ALPIM})
\text{GO TO } 720
\text{CALL YZERO}(X,Y,\text{ALPREF},\text{ALPIM})
\text{GO TO } 720
\text{FALPHA}=\text{PI} \times \text{ALPHA}
COX = COS(PALPHA)
SIX = SIN(PALPHA)
EXY = EXP(PI*BETA)
EXY1 = 1.0/EXY
COSH = 0.5*(EXY+EXY1)
SINH = 0.5*(EXY-EXY1)
DEN = (SIX*COSH)*(SIX*COSH) + (COX*SINH)*(COX*SINH)
ERE = (SIX*COX)/DEN
EIM = (-COSH*SINH)/DEN
ABSQ3 = 2.0*(ALPHA*ALPHA+BETA*BETA)
ALPRE = ERE - (((QRE*ALPHA+BETA*QIM)/ABSQ3)
ALPIM = EIM - (((QIM*ALPHA-BETA*QRE)/ABSQ3)
YRE(1) = ALPRE*BJRE(1) - ALPIM*BJIM(1) + DRE
YIM(1) = ALPIM*BJRE(1) + ALPRE*BJIM(1) + DIM
RETURN
END

SUBROUTINE YRECUR(X, Y, N, BJRE, BJIM, YRE, YIM)
DIMENSION BJRE(100), BJIM(100), YRE(50), YIM(50)

SSQ = X*X + Y*Y
TPI = 2.0/3.141592654
AZRE = (TPI*X)/SSQ
AZIM = (-TPI*Y)/SSQ
L = N + 1.
DO 1 I = 3, L
ZRE = BJRE(I)*YRE(I-1) - BJIM(I)*YIM(I-1)
ZIM = BJIM(I)*YRE(I-1) + BJRE(I)*YIM(I-1)
BZR E = ZRE - AZRE
BZIM = ZIM - AZIM
BJSQ = BJRE(I-1)*BJRE(I-1) + BJIM(I-1)*BJIM(I-1)
CZRE = BJRE(I-1)/BJSQ
CZIM = (-BJIM(I-1))/BJSQ
YRE(I) = BZRE*CZRE-BZIM*CZIM
YIM(I) = BZIM*CZRE+BZRE*CZIM
1 RETURN
END

SUBROUTINE YZERO(X, Y, ALPRE, ALPIM)
COMPLEX R, Z
TPI = 2.0/3.141592654
Z = CMPLX(X, Y)
R = CLOG(Z)
A = REAL(R)
B = AIMAG(R)
ALPRE = TPI*(-.1159315157+A)
ALPIM = TPI*B
RETURN
END

SUBROUTINE FACTOR(X, Y, ALPHA, BETA, Q, R)
COMPLEX A, P, W, Z, CLOG, CEXP
Z = CMPLX(X, Y)
A = CMPLX(ALPHA, BETA)
CALL LOGGAM(ALPHA+1.0, BETA, U, V)
W = CMPLX(U, V)
F = CLOG(Z)
IF( AIMAG(P)) 1, 2, 2
1 P=P+(0.0, 6.2831853071)
2 P=CEXP(W+ (.693147181, 0.)*A-A*P)
Q=REAL(P)
R=AIMAG(P)
RETURN
END
SUBROUTINE LOGGAM(X,Y,U,V)
DIMENSION H(7)
C LOGARITHM OF GAMMA FUNCTION
C X, Y ARE REAL AND IMAGINARY ARGUMENT
C U, V FUNCTION VALUES
B1=0.0
B2=0.0
H(1)=2.269488974
H(2)=1.517473649
H(3)=1.011523068
H(4)=5.256064690E-01
H(5)=2.523809524E-01
H(6)=3.333333333E-02
H(7)=8.333333333E-02
E2=1.57079632679
E8=3.14159265359
J=2
X2=X
3797 IF(X)<2.794, 2.793, 100
100 B6=ATAN(Y/X)
T=X*X
5793 B7=Y*T+T
C REAL PART OF LOG
T1=.5*ALOG(B7)
IF(X-2.0)1, 3793, 3793
1 B1=B1+B6
B2=B2+T1
X=X+1.0
J=1
GO TO 3797
3793 T3=-Y*B6+(T1*(X-.5)-X+9.189385332E-01)
T2=B6*(X-.5)+Y*T1-Y
T4=X
T5=Y
T1=B7
DO 200 I=1,7
T=H(I)/T1
T4=T*T4+X
T5=-T*T5+Y
200 T1=T4*T4+T5*T5
T3=T4-X+T3
T2=-T5-Y+T2
GO TO (8795, 4794), J
8795 T3=T3-B2
T2=T2-B1
4794 IF(X2)<4.796, 4.795, 4.795
4795 U=T3
V=T2
X=X2
RETURN

C
X IS ZERO
2793
T=0.0
IF(Y)5790,9999,5791
5790
B6=-E2
GO TO 5793
5791
B6=E2
GO TO 5793
C
C
X IS NEGATIVE
C
2794
E4=0.0
E5=0.0
IE6=0.
5797
E4=E4+.5*(ALOG(X*X+Y*Y))
E5=E5+ATAN(Y/X)
IE6=IE6+1
X=X+1.0
2
IF(X)5797,2,2
3
IF(MOD(IE6,2))3,3797,3
3797
E5=E5+E8
GO TO 3797
9999
WRITE(1,9990)X2,Y
STOP
9990
FORMAT( '1 ATTEMPTED TO TAKE LOGGAM OF X=',F6.0,'1X, 'Y=',F6.0)
END
SUBROUTINE START1(X,Y,N,K,R)
SSQ=X*X+Y*Y
XTEN=SQR(T(SSQ))+20.0
NTEN=IABS(N)+10.
KTEN=XTEN
M=MAXO(KTEN,NTEN)
M=M/2
K=2.*M+1
R=2.*M+2
RETURN
END

**
*** This subroutines computes the values of the modified Besel functions of the first and second kind as well as their first and second derivatives.
**
** KN = Modified Bessel of first kind
**
** IN = Modified Bessel of second kind
**
** DKN, DIN = First derivatives of KN and IN
**
** D2KN, D2IN = Second derivatives of KN and IN
** N = Order of Bessel function **
** X and Y = Real, Imaginary part of the argument **

SUBROUTINE KAI(KN,DKN,D2KN,IN,DIN,D2IN,N,X,Y)
COMPLEX KN,DKN,D2KN,IN,DIN,D2IN,Z,K1,K2,IO

XN=FLOAT(N)
Z=CMP LX{X,Y)
CALL IM(N,IN,Z)
IF(N. NE. 0)GO TO 10
CALL IM(1,DIN,Z)
D2IN=IN-DIN/Z
GO TO 100
10 CALL IM(N-1,IO,Z)
DIN=IO-XN*IN/Z
D2IN=(XN**2+XN+Z**2)*IN/Z**2-IO/Z
100 IF(N. NE. 0)GO TO 200
CALL KM(N+1,K1,Z)
CALL KM(N+2,K2,Z)
KN=K2-2.*K1/Z
DKN=-K1
D2KN=K2-K1/Z
RETURN
200 CALL KM(N,KN,Z)
IF(N. NE. 1)GO TO 210
CALL KM(N+1,K2,Z)
DKN=KN/Z-K2
D2KN=KN+K2/Z
RETURN
210 CALL KM(N-1,K1,Z)
DKN=-K1-XN*KN/Z
D2KN=(XN**2+Z**2+XN)*KN/Z**2+K1/Z
RETURN
END
SUBROUTINE PH(N,PHIN)
PHIN=0.0
IF(N. NE. 0)GO TO 10
RETURN
10 DO 20 I=1,N
20 PHIN=PHIN+1./FLOAT(I)
RETURN
END
SUBROUTINE FACT(N,FACTO)
IF(N. NE. 0)GO TO 10
FACTO=1.0
RETURN
10 FACTO=1.0
DO 20 I=1,N
20 FACTO=FACTO*FLOAT(I)
RETURN
END
SUBROUTINE GAMMFF(N,GAMMA)
CALL FACT(N-1,GAMMA)
SUBROUTINE IM(N, IN, Z)
COMPLEX IN, Z, SUM
ER=1.0E-20
K=0
SUM=(0.0, 0.0)
5 CALL FACT(K, FACTO)
CALL GAMMF(N+K+1, GAMMA)
IN=(Z/2.)**((2*K+N)/(FACTO*GAMMA))+SUM
IF((CABS(IN)-CABS(SUM)).LT.ER)RETURN
SUM=IN
K=K+1
GO TO 5
END

SUBROUTINE KM(N, KN, Z)
COMPLEX KN, Z, IN, SUM, SUM1
EULER=0.5772156
SUM=(0.0, 0.0)
SUM1=(0.0, 0.0)
ER=1.0E-15
CALL IM(N, IN, Z)
SUM=((-1.0)**(N+1)*((CLOG(Z/2.)+EULER)*IN)
NM=N-1
DO 10 K=0, NM
CALL FACT(N-K-1, FACTO)
10 SUM1=SUM1+(-1.0)**K*FACTO*((Z/2.)**((2*K-N))
SUM=SUM+0.5*SUM1
K=0
SUM1=(0., 0.)
15 CALL PH(K, PHI1)
CALL PH(N+K, PHI0)
PHI2=PHI0+PHI1
CALL FACT(K, FAKTO)
CALL FACT(N+K, FACT2)
KN=PHI2/(FACTO*FACT2)*((Z/2.)**((N+2*K)))+SUM1
IF((CABS(KN)-CABS(SUM1)).LT.ER)GO TO 20
SUM1=KN
K=K+1
GO TO 15
20 KN=SUM+((-1.0)**N)/2.*KN
RETURN
END