Modified Dugdale Model

Prem Chandra Gupta

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MODIFIED DUGDALE MODEL

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mechanical Engineering,
South Dakota State University

1972
MODIFIED DUGDALE MODEL

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable for meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser

Date

Head, Mechanical Engineering

Date
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The author also wishes to thank Prof. J.F. Sandfort for his advice and encouragement during the graduate work.
NOTATIONS

a 
half length of the extended Dugdale crack.

\( \beta \) 
normalized applied stress, \( \beta = \frac{\pi \sigma}{2Y} \)

C 
a material constant.

E 
Young's modulus.

\( \varepsilon, \varepsilon_Y \) 
strain and its yield value.

\( \eta \) 
a numerical factor depending on the type of cyclic stress.

I(N,m) 
an integral defined by Eq. 3.4 and Eq. 3.5

J(N,m) 
an integral defined by Eq. 4.4 and Eq. 4.5

K 
stress intensity factor.

\( \Delta K \) 
stress intensity range.

\( K_c \) 
static fracture toughness.

l 
half crack length.

\( l_* \) 
characteristic structural length, \( l_* = \frac{K_c^2}{2 \pi Y^2} \)

\( \frac{\Delta l}{\Delta n} \) 
rate of fatigue crack propagation.

L(N,m) 
an integral defined by Eq. 5.4 and Eq. 5.5

m 
a parameter, \( m = \frac{l}{a} \)

M 
fatigue crack propagation exponent.

n 
number of cycles.

N 
strain hardening exponent.

p(u) 
pressure along the crack surface.

\( \Pi \) 
normalized plastic work rate
\( R \)  plastic zone size.

\( \sigma \)  nominal applied stress.

\( S(x) \)  restraining stress acting at the end section of the crack.

\( t \)  coordinate indicating distance from the crack center.

\( U_y(x, 0) \) or \( U_y(x, l) \) displacement within the crack plane.

\( u \)  a dimensionless variable.

\( \bar{U} \)  normalized displacement within the crack plane.

\( U_c \)  critical displacement.

\( \mathcal{S}_{wp} \)  plastic work lost within the end section of the crack.

\( \frac{\mathcal{S}_{wp}}{\mathcal{G}_A} \)  rate of plastic work.

\( x \)  coordinate indicating distance from the crack center, \( x = \frac{X}{a} \)

\( X \)  coordinate indicating distance from the crack center.

\( Y \)  yield point.
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Elastic-plastic and plastic analyses of the stress fields in the vicinity of the crack tip are essential for the better understanding of fracture. In contrast to elastic fracture mechanics methods of analysis in the inelastic range are much less developed. For a given geometry and loading configuration for an elastic system containing cracks, the stress and displacement distribution and hence the condition for instability is uniquely determined. For an inelastic material the stress distribution depends on the history of loading. Thus, the results presented for the inelastic part are, in places, tentative and incomplete.

A precise determination of the influence of plastic yielding on the deformation and failure at a crack tip is needed for accurate predictions of the behavior of cracked bodies under static load large enough to cause fracture, and subcritical repetitive loads which cause fatigue. The catastrophic growth of cracks in plates under monotonically increasing load depends on a number of factors. A general view concerning the behavior of material at the leading edge of a crack is that plastic flow and subsequent fracture of the material is influenced by factors such as strain hardening, strain rate, the state of stress, and temperature. Therefore, for a proper understanding of fracture mechanism it is essential to determine appropriate inelastic solutions describing the behavior of plastic zone as a function of loads and mechanical properties.
The elastic-plastic problem has been considered by Rosenfield et al. (1) and Swedlow (2). Rosenfield has described experiments revealing the shapes and extent of the plastic zone in front of notches and cracks. While these analyses have produced information to indicate the role of plasticity in notched or cracked plates, they have yet to produce a criterion for fracture useful to the designer.

Dugdale model for static yielding at the tip of the crack has been considered by Newman (3) and extended to include the influence of stress-strain curve on the plastic zone size and subsequently on the fracture strength of the plate. A fracture toughness equation which accounts for non-perfect plasticity has been derived by him for the uniformly loaded plate.

Elastic-plastic crack problems in plane strain and plane stress have been discussed by Rice (4) through the use of deformation and incremental plasticity theories. The analysis is applicable to small scale yielding only and most of the results are approximate. Nevertheless, his treatment provides an insight to the inelastic fracture.

In the present work, Dugdale model (5) for static yielding at the crack tip has been used and then modified to include the effect of strain-hardening on the plastic zone size and crack tip displacement. An attempt has been made to determine the plastic energy dissipation for the strain hardening material. These properties of material are influenced by factors such as the state of stress (e.g. plane stress
or plane strain), strain rate, and the test temperature. In the present work, only the case of plane stress at room temperature for rate-insensitive material is considered.

Many attempts have been made in recent years to establish a satisfactory relationship between engineering design parameters and cyclic crack growth rates. Generally, these proposed relationships indicate that the fatigue crack growth rate is dependent upon the alternating stress level and the current crack length. This consideration has led Paris (6) to develop a relatively simple power function which provides an empirical relationship between the rate of fatigue crack growth and the corresponding stress intensity factor. This relationship is expressed as:

\[
\frac{dl}{dn} = C \cdot \Delta K^M
\]

Where \( \frac{dl}{dn} \) is the crack propagation rate; \( C \) is a constant which depends upon the material, the relative mean load, and the frequency; and \( \Delta K \) is the stress intensity range. The value of the exponent \( M \) as first determined by Paris was 4; however, recent investigations indicate that \( M \) can vary from 1 to 6 depending upon the material and stress level (7). Both \( M \) and \( C \) have to be determined experimentally.

A Law of Fatigue crack propagation has also been derived by Wnuk (8). For small stress intensity range, this law is almost identical with the power law for high-cycle fatigue as developed by Paris. In the
present paper, a rather simple approach has been used to develop the fatigue crack propagation law for a small stress intensity range (i.e. for high cycle fatigue).

The main objective of the present work was to modify the Dugdale model to include the influence of strain hardening on the plastic zone size and crack tip displacement. For a given stress level, both plastic zone size and tip displacement decrease with increasing strain hardening. For the non-hardening case, the modified Dugdale model converges very well to the original Dugdale model.

2.1 Plastic Zone Size

In Griffith's theory, the idealized media remain linearly elastic as the crack extends. There is no expectation that they can represent cracks in normally ductile materials such as structural steels.

Dugdale (5) considered the case of plane stress applied to a crack in a thin metallic sheet. This model has a wedge-shaped plastic zone ahead of the crack tip as shown in Fig. 1(a). The plastic zone may be replaced by an internal stress distribution acting on the boundary of the plastic zone as shown in Fig. 1(b). The model is based on the following postulate:

1. The material in the thin plastic zone is under a uniform tensile yield stress $\sigma$. 
2.1 Plastic Approach

Plasticity analysis aspects of cracks are much less advanced than the elastic study of the subject. However, simple aspects can be treated in ways which permit an understanding of the brittle-ductile fracture mode transition and provide essential guidance for studies of the relationship of fracture toughness and plastic flow properties. Attempts to provide general characterization factors for crack extension in terms of theory of plasticity analysis (rather than linear analysis) have been only partially successful.

2.2 Plastic Zone Size

In Griffith's theory, the idealized media remain linearly elastic as the crack extends. There is no expectation that they can represent cracks in normally ductile materials such as structural metals.

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1. The material in the thin plastic zone is under a uniform tensile yield stress $\gamma$. 
2. The transverse dimension of the plastic zone is so small that the elastic region outside may be regarded as bounded internally by a flattened ellipse of length $2(l + R)$, as in Fig. 1(a).

3. The length $R$ of the plastic zone is such that there is no stress singularity at the ends of the flattened ellipse.

These postulates are close to those adopted by Barenblatt (9), who used it to study the cohesive strength of brittle materials. In Dugdale model the yielding is assumed to be confined to a narrow zone directly ahead of the crack tip. This model can be also viewed as making the crack longer by an amount equal to the plastic zone size $R$, with yield (or cohesive) stresses acting on the extended crack surface so as to restrain the opening. Using such a model, Dugdale obtained the following solution for the plastic zone size:

$$R = l \left( \sec \frac{\pi \sigma}{2Y} - 1 \right)$$  \hspace{1cm} (2.1)

In terms of normalized plastic zone size $R/l$ and applied stress $\beta = \frac{\pi \sigma}{2Y}$, the above expression can be re-written as:

$$\frac{R}{l} = \sec \beta - 1$$  \hspace{1cm} (2.2)

A plot of normalized plastic zone size as a function of normalized applied stress is shown in Fig. 2. It can be seen that for small scale yielding, the plastic zone size increases slowly with the applied stress. But at large scale yielding, the plastic zone
size increases rapidly with applied stress, approaching infinity at yield stress. As the applied stress \( \sigma \) approaches the yield stress \( \gamma \) (i.e. at \( \frac{\sigma}{2} \)), the whole material starts yielding uniformly and this explains the sharp increase in the size of the plastic zone ahead of the crack.

2.3 Crack Tip Displacement

Dugdale model ignores the sheet thickness, and therefore the analysis model plastic zone is simply a line segment extending ahead of the apparent tip of the crack. Concentration of all the plastic strain into a line results in a representation which permits calculations of the opening displacement discontinuity. A mathematical method developed by Muskhelishvili was used by Goodier and Field (10) to solve for the crack boundary displacements for Dugdale model. According to this model, the displacement at any point in the crack plane is given by

\[
U_y(x) = \frac{2lY}{\pi E} \left[ x \ln \left( \frac{\sqrt{1-m^2} - \sqrt{1-m^2x^2}}{\sqrt{1-m^2} + \sqrt{1-m^2x^2}} \right) \right]
\]

where \( m = \frac{l}{a} \). The displacement at the tip of the actual crack in the model is

\[
U = \frac{4lY \log \sec \frac{\pi \sigma}{2\gamma}}{\pi E}
\]
Introducing the normalized displacement and applied stress, the above relation can be rewritten as

$$\bar{U} = \log \sec \beta$$  \hspace{1cm} (2.5)

where $\bar{U} = \frac{U}{4Y/\pi E}$. Plot of normalized crack tip displacement as a function of applied stress is shown in Fig. 2. The tip displacement increases rapidly for large scale yielding. As the applied stress approaches the yield stress (i.e. at $\beta = \frac{\pi}{2}$), the plastic zone spreads to the entire body, thereby making it impossible for the stress to follow the displacement. This explains the infinite size of the crack tip displacement at the yield stress.

In the case of small scale yielding, the normalized plastic zone size and crack tip displacement can be approximated as follows:

$$\frac{R}{L} \sim \frac{\beta^2}{2}$$  \hspace{1cm} (2.6)

and

$$\bar{U} \sim \frac{\beta^2}{2}$$  \hspace{1cm} (2.7)

Both the above relations resulted from McLaurin's expansions of Eqs. (2.2) and (2.5). Thus, for small scale yielding, both normalized plastic zone size and normalized crack tip displacement can be approximated by $\frac{\beta^2}{2}$. A plot of small scale yielding solutions is shown in Fig. 2. It can be seen that, as expected, all the three curves coincide for the small scale yielding case.
All the above analyses of Dugdale model apply only to perfectly plastic materials. In cracked plate tests on Steels, Dugdale (5), Rosenfield et al. (1), and Forman (11) have observed a zone of plastically deformed material consistent in shape and magnitude with the wedge-shaped zone assumed in the Dugdale model. On the other hand, Ault and Spretnak (12) with sharp notches in molybdenum, and Gerberich (13) with cracks in several aluminum alloys have observed plastic zones which differ considerably from the wedge-shaped zone. Analytical works of Stimpson and Eaton (14) and Swedlow (2) indicate plastic zones more nearly in agreement with the latter observations (12) and (13). However, the simplicity of the Dugdale model allows a mathematical treatment of plastic yielding at the crack tip. One of the assumptions of the Dugdale model is that the internal stress in the plastic zone is constant. In actuality, this stress distribution is not uniform and its shape varies with the material properties.
CHAPTER III
MODIFIED DUGDALE MODEL: PLASTIC ZONE SIZE

3.1 Ramberg Osgood Law

Strain hardening implies that the ultimate strength of a material can be increased by plastic straining. Dugdale's model of plastic yielding discussed in the previous chapter does not produce satisfactory results in the case of materials which show strain hardening. In the present chapter, Dugdale model for static yielding at the tip of a crack is extended to include the effect of strain hardening on the plastic zone size. A material with strain hardening can be described by Ramberg-Osgood power law (see Fig. 3(a)).

\[
\sigma = \begin{cases} 
\frac{E\varepsilon}{Y}, & \varepsilon \leq \varepsilon_Y \\
Y\left(\varepsilon/\varepsilon_Y\right)^N, & \varepsilon \geq \varepsilon_Y
\end{cases}
\]

\[
p(u) = \begin{cases} 
\sigma, & 0 \leq u \leq m \\
\sigma - Y\left(\frac{R}{a}\right)^{N/N+1} - \frac{1}{(u-m)^{N/N+1}}, & m \leq u \leq 1
\end{cases}
\]

where \(\varepsilon_Y\) is the field strain \(Y/E\), and \(u = X/a\), \(N\) is the strain hardening exponent. Since \(R/a = (a-k)/a = 1-m\), the power law can be rewritten as

\[
p(u) = \begin{cases} 
\sigma, & 0 \leq u \leq m \\
\sigma - Y\left(1-m\right)^{N/N+1} - \frac{1}{(u-m)^{N/N+1}}, & m \leq u \leq 1
\end{cases}
\]
The second of Eqs. (3.2) is applicable for stresses in the plastic range.

3.2 Finiteness Condition.

In this section Finiteness condition is used to determine plastic zone size for a strain hardening material. According to Finiteness condition, the total stress intensity factor should be zero.

\[ K_{\text{total}} = 2 \sqrt{\frac{l}{\pi}} \int_{0}^{1} \frac{p(u)du}{\sqrt{1-u^2}} = 0 \quad 3.3 \]

Substitution of stress distribution \( p(u) \) into Eq. (3.3) from Eq. (3.2) gives

\[ (\pi/2)\sigma - Y (1-m)^{N/N+1} \int_{m}^{1} \frac{du}{(u-m)^{N/N+1} \sqrt{1-u^2}} = 0 \quad 3.4 \]

If we denote the integral \( \int_{m}^{1} \frac{du}{(u-m)^{N/N+1} \sqrt{1-u^2}} \) by \( I(N,m) \) then

Eq. (3.4) can be rewritten as

\[ \mathcal{B} = (1-m)^{N/N+1} I(N,m) \quad 3.5 \]

This equation gives a relation between normalized stress \( \mathcal{B} \) and the parameter \( m \). The integral \( I(N,m) \) does not have a closed form solution for an arbitrary strain hardening exponent \( N \). It, therefore, prohibits an analytical solution of Eq. (3.5) for a strain hardening material.
The integral $I(N;m)$ has been calculated numerically (see Appendix A) and a plot of the parameter $m$ as a function of normalized stress $\beta$ for different strain hardening exponent $N$ is shown in Fig. 4.

### 3.3 Plastic Zone Size

The actual plastic zone size is related to $m$ by

$$\frac{R}{\ell} = a \frac{l}{\ell} = \frac{1 - m}{m}$$  \hspace{2cm} (3.6)

A replot of normalized plastic zone size $R/\ell$ as a function of normalized stress $\beta$ for different strain hardening exponent $N$ is shown in Fig. 5. It can be seen that for strain hardening material, the increase in plastic zone size is not as rapid as for perfectly plastic material (Dugdale model). In other words, for the same stress $\beta$ the plastic zone size $R/\ell$ decreases with increasing strain hardening exponent $N$. The values of strain hardening exponent $N$ were varied from 0 to 0.5. The case for the strain hardening exponent equal to zero represents perfectly plastic material (Dugdale model). The curves in Fig. 5 indicate that the amount of strain hardening has a large influence on the plastic zone size. Table 3.1 shows the percent decrease in normalized plastic zone size $R/\ell$ at two different values of stress $\beta$. It is seen that for $N=0.5$ and $\beta = 0.9$, the normalized plastic zone reduces to almost half in size, compared to perfectly plastic case. It can also be noticed that for the same strain hardening exponent $N$, the percent decrease in normalized...
plastic zone size is more at higher stress level $\beta$.

Table 3.1.

Percent decrease in normalized plastic zone size $R/l$ due to strain hardening.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\beta = 0$</th>
<th>$\beta = 0.1$</th>
<th>$\beta = 0.2$</th>
<th>$\beta = 0.3$</th>
<th>$\beta = 0.4$</th>
<th>$\beta = 0.5$</th>
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<tbody>
<tr>
<td>0.6</td>
<td>0</td>
<td>13.6</td>
<td>22.7</td>
<td>31.8</td>
<td>38.6</td>
<td>45.4</td>
</tr>
<tr>
<td>0.9</td>
<td>0</td>
<td>15.9</td>
<td>28.6</td>
<td>38.1</td>
<td>46.0</td>
<td>48.4</td>
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CHAPTER IV

MODIFIED DUGDALE MODEL
TIP DISPLACEMENT AND PLASTIC WORK RATE

4.1 Having described the effect of strain-hardening on the plastic zone size in the previous chapter, it is now our objective to do the same for crack tip displacement in strain hardening material under plane stress. Measurement of the plastic displacement at the crack tip in a strain hardening material is fundamental for understanding the behavior of a defect in a solid. Theoretical and experimental work has been done by Rosenfield et al. (1) on strain hardening materials. Their experiments show that the predictions of the modified Dugdale model were reasonably accurate. Similar work has also been done by Newman (3). In the present chapter Ramberg-Osgood power law is used to predict the crack tip displacement in a strain hardening material.

In the later part of this chapter an attempt is made to arrive at the rate of plastic work for a strain hardening material. However, due to the complexity of the problem, only the perfectly plastic case has been analyzed.

4.2 Tip Displacement

A Dugdale crack contained in a large sheet under uniform stress applied at infinity and perpendicular to the crack plane can be described by the superposition of two stress states, as shown in Fig. 6.
Superposition of two stress states produces the desired configuration of loading: traction free crack embedded in an infinite sheet under applied stress $\sigma$ . The end sections of the crack ($l \leq x \leq a$) however, are subjected to tractions $S(x)$ which hold the edges together. As the stress state I in Fig. 6 does not produce any displacement in the crack plane, it is sufficient to consider state II. An auxiliary function $f(t)$ for the state II is first computed

$$f(t) = \int_{0}^{t} \frac{p(u) \, du}{\sqrt{t^2 - u^2}}$$

where stress distribution $p(u)$ is given by

$$p(u) = \begin{cases} 
\sigma, & 0 < u < m \\
\sigma - Y(1-m)^{N/N+1} \frac{1}{(u-m)^{N/N+1}}, & m < u < 1 
\end{cases}$$

The auxiliary function can then be written as

$$f(t) = \begin{cases} 
\int_{0}^{t} \frac{\sigma \, du}{\sqrt{t^2 - u^2}}, & 0 < t < m \\
\int_{0}^{m} \frac{\sigma \, du}{\sqrt{t^2 - u^2}} + \int_{m}^{t} \left[ \sigma - Y(1-m)^{N/N+1} \frac{1}{(u-m)^{N/N+1}} \right] \frac{du}{\sqrt{t^2 - u^2}}, & m < t < 1 
\end{cases}$$

$$= \begin{cases} 
f_1 = Y \beta, & 0 < t < m \\
f_2 = Y \beta - Y(1-m)^{N/N+1} \int_{m}^{t} \frac{du}{(u-m)^{N/N+1} \sqrt{t^2 - u^2}}, & m < t < 1 
\end{cases}$$
Displacement in terms of the auxiliary function $f(t)$ within the plane of the crack is

$$U_y(x, o) = \frac{4a}{\pi E} \int_0^1 \frac{t f(t)}{\sqrt{t^2 - x^2}} \, dt$$

for $|x| < 1$

$$= \frac{4a}{\pi E} \begin{cases} \int_0^m \frac{t f_1(t)}{\sqrt{t^2 - x^2}} \, dt + \int_m^1 \frac{t f_2(t)}{\sqrt{t^2 - x^2}} \, dt & 0 < x < m \\ \int_0^1 \frac{t f_2(t)}{\sqrt{t^2 - x^2}} \, dt & m < x < 1 \end{cases}$$

$$= \frac{4a}{\pi E} \begin{cases} Y \beta \sqrt{m^2 - x^2} + Y \int_m^1 \frac{t \beta - (1-m)^{N/N+1}}{N/N+1} \int_m^t \frac{du}{(u-m)^{N/N+1} \sqrt(t^2-u^2)} \sqrt(t^2-x^2) \, dt & 0 < x < m \\ \int_0^m \frac{t \beta - (1-m)^{N/N+1}}{N/N+1} \int_m^t \frac{du}{(u-m)^{N/N+1} \sqrt(t^2-u^2)} \sqrt(t^2-x^2) \, dt & m < x < 1 \end{cases}$$

The crack tip displacement for a strain hardening material can now be written as

$$U_{tip} = U(x=m, l) = \frac{4a Y}{\pi E} \begin{cases} \beta \sqrt{1-m^2} - (1-m)^{N/N+1} \int_m^1 \frac{t du}{(u-m)^{N/N+1} \sqrt(t^2-u^2)} & 0 < x < m \\ \int_0^m \frac{t du}{(u-m)^{N/N+1} \sqrt(t^2-u^2)} & m < x < 1 \end{cases}$$
If we denote the integral
\[ \int_{m}^{t} \left[ \int_{m}^{t} \frac{du}{(u-m)^{N/N+1} \left( t^{2-u^{2}} \right) \left( t^{2-m^{2}} \right)} \right] \, dt \]
by \( J(N,m) \), then Eq. (4.4) can be rewritten as follows

\[ U_{\text{tip}} = 4 \frac{\ell Y}{\pi E m} \left[ \beta \sqrt{1-m^2} - (1-m)^{N/N+1} J(N,m) \right] \quad 4.5 \]

The integral \( J(N,m) \) does not have a closed form solution for an arbitrary strain hardening exponent \( N \). Therefore, it is not possible to obtain an analytical solution of Eq. (4.5) for a strain hardening material. The integral \( J(N,m) \) has been calculated numerically for different values of strain hardening exponent \( N \) (see Appendix B). Introducing the normalized tip displacement \( \overline{U} \), Eq. (4.5) can be written shortly

\[ \overline{U} = \frac{1}{m} \left[ \beta \sqrt{1-m^2} - (1-m)^{N/N+1} J(N,m) \right] \quad 4.6 \]

The parameter \( m \) is related to normalized stress \( \beta \) by finiteness condition

\[ \beta = (1-m)^{N/N+1} I(N,m) \quad 3.5 \]

Using Eq. (3.5) and Eq. (4.6), the normalized tip displacement is plotted as a function of normalized stress \( \beta \) for different strain hardening exponent \( N \), and is shown in Fig. 7. It is seen that for a given normalized stress \( \beta \) the tip displacement decreases with increasing strain hardening exponent \( N \). The values of strain hardening exponent \( N \) were varied from 0 to 0.5. The case for the strain hardening exponent equal to zero represents a perfectly plastic material...
(Dugdale model). Fig. 7 shows that the amount of strain-hardening has a large influence on crack tip displacement. With increasing amount of strain hardening, the increase in tip displacement is not as rapid as for perfectly plastic material (Dugdale model). Table 4.1 shows the percent decrease in normalized tip displacement $\bar{U}$ due to strain hardening at two different values of the stress $\bar{\sigma}$. It is seen that for $N=0.5$ and $\bar{\sigma}=0.9$, the normalized tip displacement decreases almost by 70%, compared to perfectly plastic case. It is also seen that for the same strain hardening exponent $N$, the percentage decrease in normalized tip displacement is greater at the higher stress level $\bar{\sigma}$.

<table>
<thead>
<tr>
<th>$\bar{\sigma}$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
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<tbody>
<tr>
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<td>0</td>
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<td>33.3</td>
<td>43.6</td>
<td>56.4</td>
<td>64.1</td>
</tr>
<tr>
<td>0.9</td>
<td>0</td>
<td>20.8</td>
<td>36.4</td>
<td>49.0</td>
<td>60.4</td>
<td>69.8</td>
</tr>
</tbody>
</table>

4.3 Plastic Work Rate

The plastic work rate for a material containing a crack under plane stress is determined by considering a moving crack. If the crack extends slowly under constant stress $\sigma$ to a new length $l + \delta l$, then the plastic zone extends correspondingly, and becomes $a + \delta a$ (see Fig. 8(a)). The displacement $U(x, l)$ of the free crack face, continued as the
elastic-plastic interface over the plastic extension becomes

\[ u(x, l + \delta l) \]. Writing \( \delta W_p \) for the work done on the plastic material, or \(-\delta W_p \) for the work done on the elastic region, we have, as given by Goodier and Field (15),

\[ \delta W_p = 4 \int_{\delta}^{a} S(x) \left( \frac{\partial u(x, l)}{\partial l} \delta l \right) dx \]

The plastic work rate is then

\[ \frac{\delta W_p}{\delta A} = \int_{m}^{1} S(x) \frac{\partial u(x, l)}{\partial l} dx \] 4.7

where \( \delta A = 4\delta l \), and \( S(x) \) is the restraining stress at the end section of the crack as shown in Fig. 6. This stress is given by

\[ S(x) = \frac{Y(1-m)^{N/N+1}}{(x-m)^{N/N+1}} \] 4.8

Of course for a perfectly plastic material \((N=0)\) we recover

\[ S(x) = -Y \] 4.8a

The displacement \( u(x, l) \) of the free crack face can then be written from the second of the Eq. (4.3),

\[ u(x, l) = \frac{4}{\pi l^2} \frac{Y}{m} \left[ \sqrt{1-x^2} - (1-m)^{N/N+1} \right] \int_{x}^{1} \frac{tdt}{\sqrt{t^2-x^2}} \int_{m}^{t} \frac{du}{(u-m)^{N/N+1}} \frac{t}{\sqrt{(t^2-u^2)}} \] 4.9
Or, for \( N=0 \)

\[
U(x, l) = \frac{4l}{\pi E m} \left[ \beta \sqrt{1-x^2} - \int x \frac{tdt}{\sqrt{t^2-x^2}} \int m \frac{du}{\sqrt{t^2-u^2}} \right], \text{ at } N=0 \quad 4.9a
\]

Using Eq. (4.7), the plastic work rate for a strain hardening material can be expressed as follows

\[
\frac{\varepsilon_{lp}}{\delta A} = \frac{-4Y^2(1-m)N/N+1}{\pi E m} \int m \frac{dx}{(x-m)^{N/N+1}} \frac{\partial}{\partial l} \left[ \int \left\{ \beta \sqrt{1-x^2} - (1-m)^{N/N+1} \right\} \right]
\]

Because of the complexity of the above equation we were unable to find a closed form solution to integrals involved. The above problem, it seems, cannot be solved analytically. It is possible to determine the plastic work rate for a particular case when strain hardening exponent \( N \) is zero (Dugdale model). In this case the plastic work rate is

\[
\frac{\varepsilon_{lp}}{\delta A} = -Y \int_m^1 \frac{\partial U(x, l)}{\partial l} \, dx
\]

Using Leibnitz rule the plastic work rate can be written as

\[
\frac{\varepsilon_{lp}}{\delta A} = Y U(l, l) + Y \frac{\partial}{\partial l} \int_m^1 U(x, l) \, dx \quad 4.11
\]

The tip displacement \( U(l, l) \) is obtained when \( x=m \).
\[ U(l, l) = \frac{4 \, l \, Y}{\pi \, E \, m} \left[ \beta \sqrt{1-m^2} - \int_{m}^{1} \frac{t \, dt}{\sqrt{t^2-m^2}} \right] \]

\[ = \frac{4 \, l \, Y}{\pi \, E \, m} \left[ \beta \sqrt{1-m^2} - \int_{m}^{1} \cos^{-1}(m) \frac{t \, dt}{\sqrt{t^2-m^2}} \right] \]

The normalized stress \( \beta \) is related to parameter \( m \) and for perfectly plastic case \( m = \cos \beta \), so that the tip displacement then becomes

\[ U(l, l) = -\frac{4 \, l \, Y}{\pi \, E} \log \cos \beta \quad \text{(4.12)} \]

Now the integral

\[ \int_{m}^{1} U(x, l) \, dx = \frac{4 \, l^2 \, Y}{\pi \, E} \frac{1}{m^2} \left\{ \beta \sqrt{1-x^2} - \int_{m}^{1} \frac{t \, dt}{\sqrt{t^2-x^2}} \right\} dx \]

\[ = \frac{4 \, l^2 \, Y}{\pi \, E \, m^2} \left\{ \beta \sqrt{1-x^2} - \int_{m}^{1} \cos^{-1}(m) \frac{t \, dt}{\sqrt{t^2-x^2}} \right\} dx \]

Finally, the plastic case (4.11)

\[ \frac{4 \, l^2 \, Y}{\pi \, E \, m^2} \left\{ \beta \sqrt{1-x^2} - \int_{m}^{1} \cos^{-1}(m) \sqrt{1-x^2} + m \right\} dx \]

\[ = \frac{4 \, l^2 \, Y}{\pi \, E \, m^2} \left\{ \beta \sqrt{1-x^2} - \cos^{-1}(m) \sqrt{1-x^2} + m \right\} dx \]
Since \( m = \cos \beta \), the above integral can be written as:

\[
\int_{m}^{1} U(x, l) \, dx = \frac{4l^2Y}{\pi E m} \int_{m}^{1} \left[ \int_{x}^{t} \frac{t^2 - x^2}{t^2 - m^2} \frac{dt}{t} \right] \, dx
\]

By changing the order of integration the integral is evaluated as follows:

\[
\int_{m}^{1} U(x, l) \, dx = \frac{4l^2Y}{\pi E m} \int_{m}^{1} \frac{dt}{t^{1/2}(t^2 - m^2)} \int_{m}^{t} \frac{t^2 - x^2}{t^2 - m^2} \, dx
\]

It can be observed here that if the crack is extending slowly, no question is raised whether the plastic work rate is sufficient to account for the starting of cracks in steel is given by the isotropic equation

\[
= \frac{2l^2Y}{\pi E} \int_{m}^{1} \left[ \frac{\pi}{2} \frac{t}{\sqrt{t^2 - m^2}} - \frac{m}{t} \frac{t}{\sqrt{t^2 - m^2}} \sin^{-1} \left( \frac{m}{t} \right) \right] \, dt
\]

The partial derivative of the above expression is then

\[
\frac{\partial}{\partial l} \int_{m}^{1} U(x, l) \, dx = \frac{4lY}{\pi E} \left[ \beta \tan \beta + 2 \log \cos \beta \right]
\]

Finally, the plastic work rate for Dugdale model results from Eq. (4.11)

\[
\frac{\delta W_p}{\delta A} = \frac{4lY^2}{\pi E} \left[ \beta \tan \beta + \log \cos \beta \right]
\]

If we introduce the normalized plastic work rate \( \Pi \) defined as

\[
\frac{\delta W_p}{\delta A} = \frac{4lY^2}{\pi E}, \text{ then } \Pi = \beta \tan \beta + \log \cos \beta
\]
A plot of the normalized work rate $\Pi$ as a function of stress $\beta$ is shown in Fig. 9. The work rate is small for small scale yielding, but increases rapidly for large scale yielding. It is worth noting here that this is the work rate if the crack is extending. There is nothing in the analysis to indicate that the crack must extend. Toward this end one would need to introduce a certain criterion of failure. An account of experimental investigations on the starting of cracks in steel is given by Mylonas (16).

It can be observed here that if the crack is extending slowly, no question arises of the availability of energy to provide the plastic work rate. Such an 'energy balance' is implicit in the field equations and boundary conditions of the elastic zone of the model. The extension process indicated in Fig. 8(a) is a gradual change of boundary loading on the edge of crack plane of the upper half plane. During this change, the total work of the boundary loadings is necessarily equal to the increase in strain energy, the work and energy being evaluated from the solution of the elastic boundary value problem. This is equivalent to the statement that all the energy made available elsewhere is expended in work done against the stress $S(x)$ on the edge of crack plane, i.e. in plastic work. None can be left over for assignment to new surface energy. This is simply a property of the Dugdale model. If account is to be taken of surface energy, a different model is needed.
5.1 Phases of Fatigue Life

Under a repeatedly applied cyclic load, fracture is produced by a load amplitude that is far below the load associated with fracture under a single monotonically increasing application. The magnitude of the cyclic amplitude necessary for fracture decreases with the increasing number of repetitions. This phenomenon is known as "fatigue."

It is generally accepted that in structures subjected to repeated external loads, microcracks may be nucleated very early in the fatigue life. As a result, it has been common practice to consider the fatigue life of a given part in three phases; namely, the nucleation and the propagation phases of the fatigue cracks and the final failure. Final failure is simply the fracture of the solid under a single application (i.e., last quarter-cycle) of the load. However, the distinction between the first two phases does not seem to be as clear. With due consideration to the microstructure of the medium, one may consider the crack as being a macrocrack, if it is large enough to permit the application of the motions of a homogeneous continuum. By fatigue propagation, in this chapter, we will understand the growth of macrocracks and assume that the continuum approach is applicable.

The difficulty in treating fatigue, both experimentally and theoretically, lies in the highly localized nature of the phenomenon. While this difficulty arises in every material, the less ductile the
material under the given conditions, the sharper is this localization. Thus, significant details of the fatigue mechanism may remain below the resolution of the electron microscope.

It is now well known that most of the fatigue life of a crack is spent in propagating rather than initiating a crack. A law describing the fatigue crack propagation has been derived by Paris (6). A similar relationship has also been derived by Rice (4) and Wnuk (8) and extended to include other inelastic materials (17). In the present chapter a rather simple approach has been used to derive a law of crack propagation. As it turns out this law is almost identical with the power law for high-cycle fatigue used by Paris.

5.2 Law of Fatigue Crack Propagation

Fatigue crack propagation for a crack under tensile stress may be viewed as a sequence of extensions, each of which occurs while the stress increases during the loading cycle. The area of the end section of the Dugdale crack enters in fatigue crack propagation because the plastic zone in front of the tip advances with the tip (see Fig. 10(a)). A point in the fixed coordinate system has a displacement $u_1 \approx 0$ when it first enters the zone, a displacement $u_2$ on the next cycle, and so on, until it reaches the tip at $x = L$. The sum of the displacements $\sum u_i$ accumulated during the traversal of the plastic zone by the crack front is thus the area of this zone divided by the step size $\varepsilon_l$. 


\[ \sum u_i \approx \frac{\text{Area}}{\delta l} \]

\[ = \int_a^l \frac{u(x, l)}{\delta l} \, dx \]

\[ 5.1 \]

The crack does not fail catastrophically because when the tip propagates a bit it enters a zone where \( \sum U(\text{tip}) \) is less than the critical displacement \( U_c \). For a stable fatigue crack, the sum of the displacement should, therefore, be always less than the critical value

\[ \sum u_i \leq U_c \]

\[ 5.2 \]

The fatigue crack propagation can then be written as

\[ \frac{\delta l}{\delta n} = \frac{\eta}{U_c} \int_a^l u(x, l) \, dx \]

\[ 5.3 \]

where \( n \) is the number of stress cycles. A numerical factor \( \eta \) enters in the Eq. (5.3) depending on the type of the cyclic stress (see Fig. 10 (b)). In the case of a "push-pull" type of cyclic stress, the crack propagates only during one-quarter of a cycle (i.e. from the point 1 to 2). During the remaining three-quarters of the stress cycle (i.e. from the point 2 to 5) there is no crack propagation. The factor \( \eta \) in this case is therefore equal to 1. In the case of a "pull-pull" type of cyclic stress, \( \eta \) is equal to 2, since the crack propagates during one-half of the stress cycle (i.e. from the point 1 to 3). Using Eq. (4.3), the integral involved in Eq. (5.3) can be
written as

\[
\int_a^b U(x, \ell) \, dx = \frac{4 \ell^2 Y}{\pi \varepsilon m^2} \int_m^1 \left[ \beta \sqrt{1-m^2} \right. - (1-m)^{N/N+1} \int_m^1 \left. \right] \, dx.
\]

5.4

Because of the complexity of the problem, we were unable to find an analytical solution to the above equation. If we denote the integral

\[
\int_m^1 \left( \frac{t \, dt}{t^2-x^2} \int_m^t \frac{du}{(u-m)^{N/N+1} \sqrt{t^2-u^2}} \right) \, dx
\]

by \( L(N,m) \), then the Eq. (5.4) can be written as

\[
\int_a^b U(x, \ell) \, dx = \frac{4 \ell^2 Y}{\pi \varepsilon m^2} \left[ \beta \sqrt{1-m^2} \int_m^1 \right. - (1-m)^{N/N+1} L(N,m) \left. \right] \]

5.5

The integral \( L(N,m) \) has been computed numerically (see Appendix C).

For the case of a perfectly plastic material, the integral involved in Eq. (5.3) can be obtained from Eq. (4.13)

\[
\int_a^b U(x, \ell) \, dx = \frac{2 \ell^2 Y}{E} \left[ \beta \tan \beta + 2 \log \cos \beta \right]
\]

5.6

A plot of normalized rate of fatigue crack propagation vs. applied stress \( \beta \) is shown in Fig. 11. A small difference in exact and
numerically calculated curves for the perfectly plastic material is probably due to approximations involved in the numerical integration of $L(N,m)$. It is seen from Fig. 11 that for a given applied stress, the rate of fatigue crack propagation is small for high strain hardening material. The curves in Fig. 11 show that strain hardening slows down the fatigue.

For small values of applied stress $\beta$, the Eq. (5.6) for perfectly plastic material can be written as

\[
\int_{x} u(x,l) \, dx = \frac{2 l^2 Y}{\pi E} \left[ \beta \beta + \beta^{1/3} (\beta)^3 + \ldots + 2 \log (1-1/2 \beta^2 + \ldots) \right]
\]

\[
= \frac{2 l^2 Y}{\pi E} \left[ \beta^2 + 1/3 \beta^4 + \ldots - \beta^2 - 1/4 \beta^4 - \ldots \right]
\]

\[
= \frac{l^2 Y}{6 \pi E} \beta^4 \tag{5.7}
\]

Using this value of the area for small stresses, the law of fatigue crack propagation can be written as

\[
\frac{\varepsilon l}{\varepsilon n} = \frac{\eta l^2 Y^2}{3 \pi K_c^2} \beta^4 \tag{5.8}
\]

where critical displacement $u_c = \frac{K_c^2}{2EY}$. The applied stress $\beta$ is related to the stress intensity factor $K$ ($\beta = \frac{K}{2Y \sqrt{\pi a}}$), while the yield stress $Y$ and the fracture toughness $K_c$ can be related by the characteristic structural length $l_*$ ($K_c^2 = 2\pi l_* Y^2$). The law of fatigue crack propagation now becomes
Here we notice that the stress intensity factor K should really be replaced by the stress intensity range \( \Delta K \) since the crack is under cyclic stress. The Eq. (5.9) can, therefore, be rewritten as

\[
\frac{\delta l}{\delta n} = \eta \frac{\pi^2 \xi}{24} \left( \frac{K}{K_c} \right)^4
\]

Eq. (5.10) constitutes a law of fatigue crack propagation. It is similar to Paris fourth power law. A plot of \( \ln \left( \frac{\delta l}{\eta \pi^2 \xi} \right) \) vs. \( \ln \left( \frac{\Delta K}{K_c} \right) \) is shown in Fig. 12. Interestingly, the slope of this curve is 4, just as it was found experimentally by Paris.

The law of fatigue crack propagation developed above is applicable for small stress intensity range (i.e. for high cycle fatigue). At low values of stress intensity ranges, the rate of crack propagation increases with \( \Delta K \) rather slowly, probably as a result of the fact that the difference between the driving elastic strain energy and the resisting plastic energy remains small.

The preceding analysis shows that the rate of fatigue crack propagation is a fourth power function of stress intensity range, but in general it is not so. For deriving the law of fatigue crack propagation, it is necessary to establish the values for the various
energy components by analytical or experimental methods and to consider the crack propagation mechanism. However, the present analysis gives an insight into the mechanism, and an average value of slope = 4 in Fig. 12 has been established for a variety of materials (body-centered cubic and face centered cubic metal with various alloy contents), supporting the earlier assumption that the phenomenon of crack propagation may take place on the continuum level.
CHAPTER VI

CONCLUSIONS

1. The modified Dugdale model applies only to a rate-insensitive, strain hardening material under plane stress condition.

2. At a given applied stress, a material with higher strain hardening has a smaller plastic zone size. Presence of strain hardening reduces the rate of increase in the plastic zone size.

3. At a given applied stress, a material with higher strain hardening has a smaller tip displacement. The decrease in tip displacement due to strain hardening is more than that in plastic zone size. Strain hardening reduces the rate of increase of tip displacement.

4. Rate of plastic work increases slowly with the applied stress at small scale yielding. But at large scale yielding, it increases rapidly and approaches infinity when applied stress approaches the yield stress.

5. For low stress intensity range, the rate of fatigue-crack propagation is described by a simple fourth power function of the stress intensity range. This result is based on an assumption that the crack propagation phenomenon takes place on a macroscopic (or continuum) level.

6. Presence of strain hardening in a material slows down the rate of fatigue crack propagation.
REFERENCES


APPENDIX A

Numerical Integration of $I(N,m)$

$$I(N,m) = \int_{m}^{1} \frac{du}{(u-m)^{N/N+1} \sqrt{1-u^2}}$$

Integral $I(N,m)$ has been computed numerically by using the subroutine DQG32 on IBM 360. The subroutine performs the integration of a given function by Gaussian quadrature formula.

To compute:

$$Y = \int_{a}^{b} f(x) \, dx$$

Gaussian quadrature formula with $n = 32$ points are used.
START

READ
P(I), I=1,7

READ
Q(J), J=1,7

DO 1
I=1,7

XU=1.

XL=XNN

DO 1
J=1,7

XN=P(I)

XNN=Q(J)

CALL SUBROUTINE
DQG 32

WRITE
XN, XNN, INTG

STOP
SUBROUTINE DQG32

PURPOSE
TO COMPUTE INTEGRAL(FCT(X)), SUMMED OVER X FROM XL TO XU

USAGE
CALL DQG32 (XL,XU,FCT,Y)
PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT

DESCRIPTION OF PARAMETERS
XL - DOUBLE PRECISION LOWER BOUND OF THE INTERVAL.
XU - DOUBLE PRECISION UPPER BOUND OF THE INTERVAL.
FCT - THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION
SUBPROGRAM USED.
Y - THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.

REMARKS
NONE

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)
MUST BE FURNISHED BY THE USER.

METHOD
EVALUATION IS DONE BY MEANS OF 32-POINT GAUSS QUADRATURE
FORMULA, WHICH INTEGRATES POLYNOMIALS UP TO DEGREE 63
EXACTLY. FOR REFERENCE, SEE
V.I.KRYLOV, APPROXIMATE CALCULATION OF INTEGRALS,

SUBROUTINE DQG32(XL,XU,FCT,Y)

DOUBLE PRECISION XL,XU,Y,A,B,C,FCT

A=.5D0*(XU+XL)
B=XU-XL
C=4985319309247407800*D
Y=.35093650473564830-2*(FCT(A+C)+FCT(A-C))
C=.92805755772534700*D
Y=Y*.8137197355128350-2*(FCT(A+C)+FCT(A-C))
C=.823611277937532200*D
Y=Y+.1269603265429300-1*(FCT(A+C)+FCT(A-C))
C=.4674593037988098400*D
Y=Y+.171369314565107170-1*(FCT(A+C)+FCT(A-C))
C = .4481605773830260660*Y
Y = Y + .2141794011113340D - 1 * (FCT(A+C) + FCT(A-C))
C = .424638661662649900*Y
Y = Y + .25499026311850860D - 1 * (FCT(A+C) + FCT(A-C))
C = .397241891839712000*Y
Y = Y + .29342045738267774D - 1 * (FCT(A+C) + FCT(A-C))
C = .366910590170448450*Y
Y = Y + .32911111316180923D - 1 * (FCT(A+C) + FCT(A-C))
C = .331522133465107600*Y
Y = Y + .36172697054424253D - 1 * (FCT(A+C) + FCT(A-C))
C = .29385707652611600*Y
Y = Y + .39696947893535153D - 1 * (FCT(A+C) + FCT(A-C))
C = .253449945466114700*Y
Y = Y + .41655962113473378D - 1 * (FCT(A+C) + FCT(A-C))
C = .215675638065317670*Y
Y = Y + .43826045502201966D - 1 * (FCT(A+C) + FCT(A-C))
C = .165934301141063820*Y
Y = Y + .45566693937347881942D - 1 * (FCT(A+C) + FCT(A-C))
C = .1196436811262665400*Y
Y = Y + .4692219540402283D - 1 * (FCT(A+C) + FCT(A-C))
C = .7223598073136425D - 1 * B
Y = Y + .4781936039637430D - 1 * (FCT(A+C) + FCT(A-C))
C = .241536326438691580 - 1 * B
Y = B * (Y + .46273044257363900D - 1 * (FCT(A+C) + FCT(A-C)))
RETURN
END
APPENDIX B

Numerical Integration of J(N, m)

\[
J(N, m) = \int_{m}^{1} \left[ \int_{m}^{t} \frac{dt}{(u-m)^{N/N+1} \sqrt{t^2 - u^2}} \right] \frac{tdt}{\sqrt{t^2 - m^2}}
\]

By changing the order of integration (see Fig. 8(b)), the integral \(J(N, m)\) can be written as

\[
J(N, m) = \int_{m}^{1} \left[ \int_{u}^{1} \frac{tdt}{(t^2-u^2)(t^2-m^2)} \right] \frac{du}{(u-m)^{N/N+1}}
\]

\[
= \int_{m}^{1} \left[ \ln \left( \sqrt{t^2-m^2} + \sqrt{t^2-u^2} \right) \right] \frac{du}{(u-m)^{N/N+1}}
\]

\[
= \int_{m}^{1} \frac{\sqrt{1-m^2} + \sqrt{1-u^2}}{(u-m)^{N/N+1}} \ln \left( \frac{\sqrt{u^2-m^2}}{\sqrt{u^2-m^2}} \right) \, du
\]

Integral \(J(N, m)\) has been computed numerically on IBM 360. The subroutine and the flow diagram are same as for integral \(I(N, m)\) in Appendix A.
APPENDIX C

Numerical Integration of $L(N,m)$

$$L(N,m) = \int_{m}^{1} \left[ \int_{x}^{1} \frac{tdt}{\sqrt{t^2-x^2}} \right] \int_{m}^{x} \frac{du}{(u-m)^{N/N+1} \sqrt{u^2-u^2}} dx$$

By changing the order of integration and integrating w.r.t. $t$, the integral $L(N,m)$ becomes (see Appendix B)

$$L(N,m) = \int_{m}^{1} \left[ \int_{x}^{1} \frac{\ln \left( \frac{1-x^2 + 1-u^2}{\sqrt{u^2-x^2}} \right)}{(u-m)^{N/N+1}} du \right] dx \quad m \leq x \leq 1$$

By changing the order of integration again, we can write

$$L(N,m) = \int_{m}^{1} \left[ \int_{m}^{x} \ln \left( \frac{\sqrt{1-x^2} + \sqrt{1-u^2}}{\sqrt{u^2-x^2}} \right) dx \right] \frac{du}{(u-m)^{N/N+1}}$$

Here integral

$$\int_{m}^{1} \ln \left[ \frac{\sqrt{1-x^2} + \sqrt{1-u^2}}{\sqrt{u^2-x^2}} \right] dx$$

is

$$= \int_{m}^{1} \left[ \ln \left( \sqrt{1-x^2} + \sqrt{1-u^2} \right) - \frac{1}{2} \ln (u^2-x^2) \right] dx$$
\[
= \left[ x \ln \left( \sqrt{1-x^2} + \sqrt{1-u^2} \right) - \sqrt{1-x^2} + \sqrt{1-u^2} \ln 2 \right] - \left( \sqrt{1-x^2} + \sqrt{1-u^2} \right) \\
+ \sqrt{2-u^2} \ln 2 \left( \frac{\sqrt{2-u^2} \cdot \sqrt{2-x^2} + 1 - \sqrt{1-u^2} \cdot \sqrt{1-x^2}}{\sqrt{1-x^2} + \sqrt{1-u^2}} \right) - \frac{u}{2} \ln \frac{u+x}{u-x} \\
- \frac{x}{2} \ln \left( u^2-x^2 \right) - x \right] \bigg|_m^1 \\
= \sqrt{2-u^2} \ln \left[ \frac{\left( \sqrt{2-u^2} + 1 \right) \left( \sqrt{1-m^2} + \sqrt{1-u^2} \right)}{\sqrt{1-u^2} \left( \sqrt{2-u^2} \sqrt{2-m^2} + 1 - \sqrt{1-u^2} \sqrt{1-m^2} \right)} \right] \\
+ m \ln \left( \frac{\sqrt{u^2-m^2}}{\sqrt{1-m^2} + \sqrt{1-u^2}} \right) - \sqrt{1-u^2} \ln \left( \sqrt{2-m^2} + \sqrt{1-m^2} \right) \\
+ \sqrt{2-m^2} - m \\
= \emptyset (u,m)
\]

The integral \( L(N,m) \) can, therefore, be written as

\[
L(N,m) = \int_m^1 \frac{\emptyset (u,m) \, du}{(u-m)^{N/N+1}}
\]

Integral \( L(N,m) \) has been computed numerically on IBM 360. The subroutine and the flow diagram are same as for integral \( I(N,m) \) in Appendix A.
Fig. 1 Dugdale model for plastic yielding at the tip of a crack.
Fig. 2 Variation of plastic zone size and tip displacement as a function of normalized stress.

\[ \frac{R}{L} = \sec \beta - 1 \]

\[ \bar{U} = \frac{U_{\text{tip}}}{4\pi Y/\pi E} = \log \sec \beta \]
Fig. 3(a) Ramberg-Osgood Power Law.

\[ \sigma = \begin{cases} \frac{E \varepsilon}{Y} & \varepsilon \leq \varepsilon_Y \\ Y \left( \frac{\varepsilon}{\varepsilon_Y} \right)^n & \varepsilon \geq \varepsilon_Y \end{cases} \]

Fig. 3(b) Crack with modified internal-stress distribution.
Fig. 4 Effect of strain-hardening on parameter m.

\[ \beta = (1-m)^{\frac{N}{N+1}} I(N,m) \]

\[ I = \int_{m}^{1} \frac{du}{(u-m)^{\frac{N}{N+1}} \sqrt{1-u^2}} \]
\[ \beta = (1-m) \frac{N}{N+1} I(N,m) \]
\[ \frac{\ell}{\lambda} = \frac{1-m}{m} \]
\[ I = \int \frac{du}{m (u-m)^{N+1} \sqrt{1-u^2}} \]

**Fig. 5** Effect of strain-hardening on plastic zone size.
Fig. 6 Traction-free crack in an infinite sheet.
\[ U = \frac{(1-m)^{N+1}}{m} \left[ \int \sqrt{1-m^2} \ I - J \right] \]
\[ \beta = (1-m)^{\frac{N}{N+1}} \ I \]
\[ I = \int_{m}^{1} \frac{du}{(u-m)^{\frac{N}{N+1}}} \sqrt{1-u^2} \]
\[ J = \int_{m}^{1} \ln \left[ \frac{\sqrt{1-u^2} + \sqrt{1-m^2}}{\sqrt{u^2-m^2}} \right] \frac{du}{(u-m)^{\frac{N}{N+1}}} \]

Fig. 7 Effect of strain-hardening on crack tip displacement.
Fig. 8(a) Extending crack tip displacement.

Fig. 8(b) Area of integration for integral $J(n,m)$. 
Fig. 9 Variation of rate of plastic work with applied stress for perfectly plastic material.
Fig. 10(a) Fatigue crack propagation.

Fig. 10(b) Types of cyclic stress.
\[ \frac{1}{C_o} \frac{\delta I}{\delta \eta} = \frac{(1-m)^{n+1}}{m^2} \left[ (1-m) \int_{1-m^2}^{1} I'(N,m) - L(N,m) \right] \]

\[ C_o = \frac{m}{U_c} \frac{l_2 Y}{\pi E} \]

\[ I(N,m) = \int_{m}^{1} \frac{du}{(u-m)^{n+1}(1-u^2)} \]

\[ L(N,m) = \int_{m}^{1} \left[ \int_{m}^{u} \frac{dt}{(t-x^2)^{n+1}} \int_{m}^{u} \frac{du}{(u-m)^{n+1}(1-u^2)} \right] dx \]

**Exact solution for** \( N = 0 \)

\[ \frac{1}{C_o} \frac{\delta I}{\delta \eta} = B \tan \beta + 2 \log \cos \beta \]

**Fig. 11.** Effect of strain hardening on rate of fatigue crack propagation.
Fig. 12. Variation of rate of crack propagation with stress intensity range for high cycle fatigue.