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THE EXISTENCE OF REVERSED DIGIT ADDENDS MODULO n

By

Lance Adair Wheeler

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mathematics, South Dakota
State University

1974

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THE EXISTENCE OF REVERSED DIGIT

ADDENDS MODULO n

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable for meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Advisor Date

Head, Mathematics Department Date

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LAW

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1. Introduction

One of the most interesting and commonly pursued branches of mathematics is the study of number theory. Its appeal lies not in its applicability, but in a fascination for the properties of the numbers themselves. It should be noted that, on occasion, problems of number theory have contributed to more pragmatic branches of mathematics and hence to a practical application. Frequently this happens without the original author's expectation or intention since he was merely remarking on what he considered to be a singular or unusual facet of the employed number system.

Such a problem is that which involves the rearrangement of the digits of a given integer. The algebra student frequently encounters problems of this nature while studying systems of equations. He is asked to find an integer such that the sum of this integer and the integer generated by reversing the order of the digits is equal to a second given integer. The student, if he progresses to the study of an actual course in number theory, will again encounter problems involving reversing the order of the digits of unknown integers, and at this time further stipulations will be added. It is this problem with which this paper will deal.

In this paper, as in most work done in number theory, we shall be using the set of integers and any alphameric symbols, unless stated otherwise, will be understood to represent integers. We shall employ the system of axioms associated with the integers; this system can be found in any elementary algebra text. In addition, we shall here introduce a further postulate which will be employed in one of the following three equivalent forms:

(1) First form of the Principle of Mathematical Induction. Any set of positive integers which contains the integer 1 and the integer k+1 whenever it contains the positive integer k, contains all positive integers.

(2) Second form of the Principle of Mathematical Induction. Any set of positive integers which contains 1 and k+1 whenever it contains the integers 1 to k inclusive, contains all positive integers.

(3) Well Ordering Principle. Every non-empty set of positive integers has a least element.

The preceding postulate is a powerful tool commonly used to establish conjectures for the whole of the integers when one knows the conjecture to be true for a finite subcollection. While we shall later employ the postulate in just this manner, its immediate consequence is a fundamental property referred to as the Division Algorithm.

<u>Theorem 1.1.</u> For any $m>0$ and integer a, there exist unique integers q and r with $0\leq r<m$ such that $a=mq+r$.

The Division Algorithm states that if we were to subtract sufficiently many multiples of n from the integer a, we would eventually be left with a remainder or residue r, which it would not be possible to reduce further under the restriction that r be positive. Consequently, it could be said that there is a connection between the integer r and a. This connection is called congruence and is further delineated by:

Definition 1.1. If n is a positive integer greater than one and n divides (a-r), we say "a is congruent to r modulo n" and we write $a \equiv r \pmod{n}$. If a is not congruent to r modulo n, we write $a \not\equiv r \pmod{n}$.

Definition 1.2. If $a = nq + r$ with $0 \leq r < n$, then r is the "least residue" of a modulo n. In general, if $a \equiv s \pmod{n}$ then s is called a "residue" of a modulo n.

The posulate of Mathematical Induction also gives us the following:

Theorem 1.2. Let $b > 1$. Then every $M > 0$ can be uniquely represented in the form

$$M = a_0 + a_1 b + a_2 b^2 + \ldots + a_{k-1} b^{k-1} + a_k b^k$$

with $a_k \neq 0$, $k \geq 0$, and $0 \leq a_i < b$ for $i \leq k$.

We shall use the preceding theorem in conjunction with the special case $b = 10$ and the standard notation

$$M = \sum_{i=0}^{k} a_i 10^i = a_0 + a_1 10 + \ldots + a_k 10^k.$$

We make the following observations:

(1) if t is a constant, $\sum_{i=0}^{k} ta_i = ta_0 + ta_1 + \ldots + ta_k$

$= t(a_0 + a_1 + \ldots + a_k)$

$= t\sum_{i=0}^{k} a_i$

(2) if $a_i=1$ for all i, $\sum_{i=1}^{k} t = \sum_{i=1}^{k} ta_i$

$= t(1 + 1 + \ldots + 1)$

$= tk$

(3) $\sum_{i=0}^{k} (a_i + b_i) = (a_0 + b_0) + (a_1 + b_1) + \ldots + (a_k + b_k)$

$= (a_0 + a_1 + \ldots + a_k) + (b_0 + b_1 + \ldots + b_k)$

$= \sum_{i=0}^{k} a_i + \sum_{i=0}^{k} b_i$

(4) $a_i=0$ for all $i<0$.

We know sufficient information to now pose the question that shall be the concern of this paper:

Let n be a positive integer. Does there exist a positive integer $M=a_0 + a_1 10^1 + \ldots + a_k 10^k$ with $a_0 \neq 0$ such that

$M + a_k + a_{k-1}10^1 + a_{k-2}10^2 + \ldots + a_1 10^{k-1} + a_0 10^k \equiv 0 \pmod{n}$?

We shall conclude the introduction by stating two elementary theorems of number theory.

Theorem 1.3. Let $M=\sum_{i=0}^{k} a_i 10^i$. Then 11 divides M if and only if

$$\sum_{i=0}^{k} a_i(-1)^i \equiv 0 \pmod{11}.$$

It shall later be shown that numbers which are the sums of addends, each of which is the reversed image of the other, are frequently divisible by 11. Consequently, this theorem will constitute a useful check.

The second theorem deals with congruences and will form a basis for the proof of the existence of an integer M and its image such that these integers satisfy the conditions of the question.

Theorem 1.4. The linear congruence $ax \equiv c \pmod{n}$ is solvable if and only if the greatest common divisor of a and n also divides c where the greatest common divisor of a and n is the largest positive integer that divides a and n, and is such that any other divisor of a and n also divides it, and it is denoted by $d=(a,n)$. Furthermore, if there are any solutions, there are precisely d incongruent solutions.

2. An Alternating Generator

Let M be any positive integer, then by Theorem 1.2 there exist integers a_0, a_1, a_2, ..., a_k such that:

(1) $0 \le a_i \le 9$, for $0 \le i \le k-1$

(2) $1 \le a_k \le 9$,

(3) $M = \sum_{i=0}^{k} a_i 10^i$.

For the purpose of this paper, we shall further assume that $1 \le a_0 \le 9$ so that $\overline{M}$ defined by

$$\overline{M} = \sum_{i=0}^{k} a_{k-i} 10^i \tag{2.1}$$

makes sense.

Letting $S = M + \overline{M}$, we have

$$\begin{aligned} S &= \sum_{i=0}^{k} a_i 10^i + \sum_{i=0}^{k} a_{k-i} 10^i \\ &= \sum_{i=0}^{k} (a_i + a_{k-i}) 10^i \\ &= (a_0 + a_k)10^0 + (a_1 + a_{k-1})10^1 + (a_2 + a_{k-2})10^2 \\ &\quad + \dots + (a_{k-1} + a_1)10^{k-1} + (a_k + a_0)10^k. \end{aligned} \tag{2.2}$$

A careful examination of the coefficients of the powers of ten in (2.2) tells us that the coefficients of 10^0 and 10^k are the same; as are the coefficients of the pairs $(10^1, 10^{k-1})$, $(10^2, 10^{k-2})$, $(10^3, 10^{k-3})$,

$(10^4,10^{k-4})$, ..., $(10^x,10^{k-x})$; where the value of x is dependent upon the number of digits in M.

If there is an even number of digits in M, then there is an even number of terms in S and k is odd so that there exists an integer x such that k=2x+1. In this case, all the coefficients of (2.2) can be paired; that is, for every term involving 10^i, $0\le i\le x$, there is another term involving 10^{k-i} having the same coefficient. Hence, we have

(2.3)
$$S=(a_0 + a_{2x+1})(10^{2x+1} + 10^0) + (a_1 + a_{2x})(10^{2x} + 10^1)$$
$$+ (a_2 + a_{2x-1})(10^{2x-1} + 10^2) +\dots +$$
$$(a_{x-1} + a_{x+2})(10^{x+2} + 10^{x-1}) + (a_x + a_{x+1})(10^{x+1} + 10^x)$$

$$=(a_0 + a_{2x+1})(10^{2x+1} + 1) + (a_1 + a_{2x})(10^{2x-1} + 1)10$$
$$+ (a_2 + a_{2x-1})(10^{2x-1} + 1)10^2 +\dots +$$
$$(a_{x-1} + a_{x+2})(10^3 + 1)10^{x-1} + (a_x + a_{x+1})(10 + 1)10^x$$

$$=\sum_{i=0}^{x}(a_i + a_{2x+1-i})(10^{2x+1-2i} + 1)10^i$$

$$=\sum_{i=0}^{x}e_i10^i(10^{2(x-i)+1} + 1)$$

where $e_i=a_i + a_{2x+1-i}$ and is such that $0\le e_i\le 18$ for $1\le i\le x$ and $2\le e_0\le 18$ because of the restrictions on a_i for $0\le i\le k$.

If there is an odd number of digits in M, then there is an odd number of terms in S and k is even so that there exists an integer x such that k=2x. In this case, all the coefficients of (2.2), with the exception of one, can be paired; that is, for every term involving 10^i, $0 \le i \le x-1$, there is another term involving 10^{k-i} having the same coefficient. The exception is the coefficient of 10^x which is unpairable. Hence, we have

$$(2.4) \quad S=(a_0 + a_{2x})(10^{2x} + 10^0) + (a_1 + a_{2x-1})(10^{2x-1} + 10) + (a_2 + a_{2x-2})(10^{2x-2} + 10^2) + \dots + (a_{x-1} + a_{x+1})(10^{x+1} + 10^{x-1}) + 2a_x 10^x$$

$$= (a_0 + a_{2x})(10^{2x} + 1) + (a_1 + a_{2x-1})(10^{2x-2} + 1)10 + (a_2 + a_{2x-2})(10^{2x-4} + 1) + \dots + (a_{x-1} + a_{x+1})(10^2 + 1)10^{x-1} + 2a_x 10^x$$

$$= \sum_{i=0}^{x} (a_i + a_{2x-i})10^i(10^{2(x-i)} + 1)$$

$$= \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$$

where $f_i = a_i + a_{2x-i}$ and is such that $0 \le f_i \le 18$ for $1 \le i \le x-1$, $0 \le f_x \le 9$, and $2 \le f_0 \le 18$ because of the restrictions on a_i for $0 \le i \le k$.

Let us now assume that we are given a number S which can be written in the form of (2.3) or (2.4). We shall show that there exist integers M and $\bar{M}$ such that $S=M+\bar{M}$, where M and $\bar{M}$ satisfy the restrictions previously given.

Suppose that $S=\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1}+1)$

where $0\leq e_i\leq 18$ for $1\leq i\leq x$ and $2\leq e_0\leq 18$.

Choose a_0 such that

(2.5) $$\text{maximum}\{0,e_0-10\}<a_0<\text{minimum}\{e_0,10\}$$

and let

(2.6) $$a_{2x+1}=e_0-a_0.$$

It is obvious from (2.5) that $0<a_0<10$ so that $1\leq a_0\leq 9$. Using (2.5) again, we see that $e_0-10<a_0<e_0$ so that $-10<a_0-e_0<0$ or $1\leq e_0-a_0\leq 9$. Hence, $1\leq a_{2x+1}\leq 9$.

If $1\leq i\leq x$, we choose a_i such that

(2.7) $$\text{maximum}\{0,e_i-9\}\leq a_i\leq\text{minimum}\{e_i,9\}$$

and let

(2.8) $$a_{2x+1-i}=e_i-a_i.$$

As before, it is obvious from (2.7) that $0\leq a_i\leq 9$ and $0\leq e_i-a_i=a_{2x+1-i}\leq 9$.

Using (2.7) and (2.8) together with our suppositions, we have

(2.9) $$S=\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1}+1)$$

$$=\sum_{i=0}^{x}(a_i + a_{2x+1-i})(10^{2x+1-i} + 10^i)$$

$$=\sum_{i=0}^{x} a_{2x+1-i}10^{2x+1-i} + \sum_{i=0}^{x} a_i 10^i$$

$$+ \sum_{i=0}^{x} a_i 10^{2x+1-i} + \sum_{i=0}^{x} a_{2x+1-i}10^i$$

$$= \sum_{i=x+1}^{2x+1} a_i 10^i + \sum_{i=0}^{x} a_i 10^i + \sum_{i=x+1}^{2x+1} a_{2x+1-i}10^i + \sum_{i=0}^{x} a_{2x+1-i}10^i$$

$$= \sum_{i=0}^{2x+1} a_i 10^i + \sum_{i=0}^{2x+1} a_{2x+1-i}10^i$$

$$= M + \overline{M}$$

where M and $\overline{M}$ satisfy the restrictions previously given.

Let us now assume that $S=\sum_{i=0}^{x} f_i 10^i(10^{2(x-i)} + 1)$

where $0\le f_i\le 18$ for $1\le i\le x-1$, $2\le f_0\le 18$, and $0\le f_x\le 9$.

Choose an a_0 such that

(2.10) $\quad \text{maximum}\{0,f_0-10\}<a_0<\text{minimum}\{f_0,10\}$

and let

(2.11) $\quad a_{2x}=f_0-a_0.$

Using (2.10) and an argument similar to that for the case where S involved the e_i, we see that $1\le a_0\le 9$, and $1\le a_{2x}=f_0-a_0\le 9$.

If $1 \leq i \leq x-1$, we choose a_i such that

(2.12) $$\text{maximum}\{0, f_i - 9\} \leq a_i \leq \text{minimum}\{f_i, 9\}$$

and let

(2.13) $$a_{2x-i} = f_i - a_i.$$

Using (2.12) and a familiar argument, we see that $0 \leq a_i \leq 9$ and $0 \leq f_i - a_i = a_{2x-i} \leq 9$.

If $i=x$, the expression for S tells us that we have $2f_x 10^x$. We let a_x be given by

(2.14) $$a_x = f_x.$$

By (2.11), (2.13), and (2.14), together with our assumptions, we see, by an argument similar to that for the case where S involves the e_i, that

(2.15) $$S = \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$$

$$= \sum_{i=0}^{x-1} (a_{2x-i} + a_i)(10^{2x-i} + 10^i) + 2a_x 10^x$$

$$= \sum_{i=0}^{x-1} a_{2x-i} 10^{2x-i} + \sum_{i=0}^{x-1} a_i 10^i + a_x 10^x$$

$$+ \sum_{i=0}^{x-1} a_{2x-i} 10^i + \sum_{i=0}^{x-1} a_i 10^{2x-i} + a_x 10^x$$

$$= \sum_{i=x+1}^{2x} a_i 10^i + \sum_{i=0}^{x} a_i 10^i + \sum_{i=x+1}^{2x} a_i 10^{2x-i} + \sum_{i=0}^{x} a_i 10^{2x-i}$$

$$= \sum_{i=0}^{2x} a_i 10^i + \sum_{i=0}^{2x} a_i 10^{2x-i}$$

$$= M + \overline{M}$$

where M and $\overline{M}$ satisfy the restrictions previously given.

The preceding discussion proves the following theorem:

Theorem 2.1. A necessary and sufficient condition for a positive integer S to be written in the form $M + \overline{M}$, where

$M = \sum_{i=0}^{k} a_i 10^i$ with $0 \le a_i \le 9$ for $1 \le i < k$ and $1 \le a_i \le 9$ for i=0 or k

and $\overline{M} = \sum_{i=0}^{k} a_{k-i} 10^i$ is

(a) $S = \sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1} + 1)$ where $k=2x+1$,

$e_i = a_i + a_{2x+1-i}$ and is such that $0 \le e_i \le 18$ for $1 \le i \le x$, and $2 \le e_0 \le 18$; or

(b) $S = \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$ where $k=2x$, $f_i = a_i + a_{2x-i}$ and is such that $0 \le f_i \le 18$ for $1 \le i \le x-1$, $0 \le f_x \le 9$, and $2 \le f_0 \le 18$.

In Tables I and II, we find a partial list of integers which can be written in the form $S = M + \overline{M}$. A careful examination of Tables I and II leads one to the following theorems:

Table I

The following is a list of numbers which are of the form

$S=\sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$ and which are less than 2000.

2	504	807	1110	1413	1716
4	505	808	1111	1414	1717
6	524	827	1130	1433	1736
8	525	828	1131	1434	1737
10	544	847	1150	1453	1756
12	545	848	1151	1454	1757
14	564	867	1170	1473	1776
16	565	868	1171	1474	1777
18	584	887	1190	1493	1796
	585	888	1191	1494	1797
202					
222	605	908	1211	1514	1817
242	606	909	1212	1515	1818
262	625	928	1231	1534	1837
282	626	929	1232	1535	1838
	645	948	1251	1554	1857
302	646	949	1252	1555	1858
303	665	968	1271	1574	1877
322	666	969	1272	1575	1878
323	685	988	1291	1594	1897
342	686	989	1292	1595	1998
343					
362	706	1009	1312	1615	1918
363	707	1010	1313	1616	1938
382	726	1029	1332	1635	1958
383	727	1030	1333	1636	1978
	746	1049	1352	1655	1998
	747	1050	1353	1656	
403	766	1069	1372	1675	
404	767	1070	1373	1676	
423	786	1089	1392	1695	
424	787	1090	1393	1696	
443					
444					
463					
464					
483					
484					

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Table II

The following are numbers which are of the form

$$S=\sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1} + 1)$$ and which are less than 20,000.

22	3003	5005	7007	9009	11,000
33	3102	5104	7106	9108	11,011
44	3113	5115	7117	9119	11,110
55	3212	5214	7216	9218	11,121
66	3223	5225	7227	9229	11,220
77	3322	5324	7326	9328	11,231
88	3333	5335	7337	9339	11,330
99	3432	5434	7436	9438	11,341
	3443	5445	7447	9449	11,440
110	3542	5544	7546	9548	11,451
121	3553	5555	7557	9559	11,550
132	3652	5654	7656	9658	11,561
143	3663	5665	7667	9669	11,660
154	3762	5764	7766	9768	11,671
165	3773	5775	7777	9779	11,770
176	3872	5874	7876	9878	11,781
187	3883	5885	7887	9889	11,880
198	3982	5984	7986	9988	11,891
	3993	5995	7997	9999	11,990
2002					
2112	4004	6006	8008	10,010	12,001
2222	4103	6105	8107	10,109	12,012
2332	4114	6116	8118	10,120	12,111
2442	4213	6215	8217	10,219	12,122
2552	4224	6226	8228	10,230	12,221
2662	4323	6325	8327	10,329	12,232
2772	4334	6336	8338	10,340	12,331
2882	4433	6435	8437	10,439	12,342
2992	4444	6446	8448	10,450	12,441
	4543	6545	8547	10,549	12,452
	4554	6556	8558	10,560	12,551
	4653	6655	8657	10,659	12,562
	4664	6666	8668	10,670	12,661
	4763	6765	8767	10,769	12,672
	4774	6776	8778	10,780	12,771
	4873	6875	8877	10,879	12,782
	4884	6886	8888	10,890	12,881
	4983	6985	8987	10,989	12,892
	4994	6996	8998		12,991

Table II continued

13,002	15,004	17,006	19,008
13,013	15,015	17,017	19,118
13,112	15,114	17,116	19,228
13,123	15,125	17,127	19,338
13,222	15,224	17,226	19,448
13,233	15,235	17,237	19,558
13,332	15,334	17,336	19,668
13,343	15,345	17,347	19,778
13,442	15,444	17,446	19,888
13,453	15,455	17,457	19,998
13,552	15,554	17,556	
13,563	15,565	17,567	
13,662	15,664	17,666	
13,673	15,675	17,677	
13,772	15,774	17,776	
13,783	15,785	17,787	
13,882	15,884	17,886	
13,893	15,895	17,897	
13,992	15,994	17,996	
14,003	16,005	18,007	
14,014	16,016	18,018	
14,113	16,115	18,117	
14,124	16,126	18,128	
14,223	16,225	18,227	
14,234	16,236	18,238	
14,333	16,335	18,337	
14,344	16,346	18,348	
14,443	16,445	18,447	
14,454	16,456	18,458	
14,553	16,555	18,557	
14,564	16,566	18,568	
14,663	16,665	18,667	
14,674	16,676	18,678	
14,773	16,775	18,777	
14,784	16,786	18,788	
14,883	16,885	18,887	
14,894	16,896	18,898	
14,993	16,995	18,997	

Theorem 2.2. If $S=\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1}+1)$ then 11 divides S.

Proof.--Since $10\equiv-1 \pmod{11}$, we have $10^{2(x-i)+1}\equiv-1^{2(x-i)+1}\equiv-1 \pmod{11}$. Hence, $S\equiv0 \pmod{11}$ and 11 divides S; for which we write $11|S$.

Theorem 2.3. Let $S=M+\overline{M}=\sum_{i=0}^{x} f_i 10^i(10^{2(x-i)}+1)$. Then there exists an $M=\overline{M}$ if and only if $2|f_i$ for $0\le i\le x-1$.

Proof.--Because of the form of S, we know that $M=\sum_{i=0}^{k} a_i 10^i$ where $k=2x$. It is now obvious that $M=\overline{M}$ if and only if $a_{k-i}=a_i$ for $0\le i\le x-1$ if and only if $f_i=2a$ for $0\le i\le x-1$.

Theorem 2.4. Let $S=M+\overline{M}=\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1}+1)$. Then there exists an $M=\overline{M}$ if and only if $2|e_i$ for $0\le i\le x$.

Proof.--Since the proof of Theorem 2.4 is identical to that of Theorem 2.3, the details have been omitted.

Theorem 2.5. If $0\le e_i+e_{i-1}\le18$ for $1\le i\le x-1$, $0\le e_x\le9$, and $2\le e_0\le18$, then

$$11\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1}+1)=\sum_{i=0}^{x+1} f_i 10^i(10^{2(x+1-i)}+1).$$

Proof.--Using the simple fact that 11=10 + 1, we have

$$11\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1} + 1)$$

$$= \sum_{i=0}^{x} e_i(10^{2x-i+2} + 10^{i+1}) + \sum_{i=0}^{x} e_i(10^{2x-i+1} + 10^i)$$

$$= \sum_{i=0}^{x} e_i 10^{2x-i+2} + \sum_{i=0}^{x} e_i 10^{i+1} + \sum_{i=0}^{x} e_i 10^{2x-i+1} + \sum_{i=0}^{x} e_i 10^i$$

$$= \sum_{i=1}^{x} e_i 10^{2x-i+2} + \sum_{i=1}^{x} e_{i-1} 10^i + \sum_{i=1}^{x} e_{i-1} 10^{2x-i+2} + \sum_{i=1}^{x} e_i 10^i$$

$$+ e_0 10^{2x+2} + e_x 10^{x+1} + e_x 10^{x+1} + e_0$$

$$= \sum_{i=1}^{x} (e_i + e_{i-1}) 10^i(10^{2(x-i+1)} + 1) + e_0(10^{2x+2} + 1) + 2e_x 10^{x+1}$$

$$= \sum_{i=0}^{x+1} f_i 10^i(10^{2(x+1-i)} + 1)$$

where $f_i = e_i + e_{i-1}$ for $1 \le i \le x$, $f_0 = e_0$, and $f_{x+1} = e_x$. Obviously, the f_i s satisfy the conditions of Part (b) of Theorem 2.1.

Table III illustrates the results of Theorem 2.5.

Table III

The entries of Table III are the result of multiplying the entries of Table II by 11. An asterisk (*) indicates that the entry cannot be written in the form S=M + M̄.

242	33,033	55,055	77,077	99,099
363	34,122*	56,144*	78,166*	100,188*
484	34,243	56,265	78,287	100,309
605	35,332*	57,354*	79,376*	101,398*
726	35,453	57,475	79,497	101,519
847	36,542*	58,564*	80,586*	102,608*
968	36,663	58,685	80,707	102,729
1089	37,752*	59,774*	81,796*	103,818*
	37,773	59,895	81,917	103,939
1210*	38,962*	60,984*	83,006*	105,028*
1331*	39,083	61,115	83,127	105,149
1452*	40,172*	62,095*	84,216*	106,238*
1573*	40,293	62,315	84,337	106,359
1694*	41,382*	63,404*	85,426*	107,448*
1815*	41,503	63,525	85,547	107,569
1936*	42,592*	64,614*	86,636*	108,658*
2057*	42,713	64,735	86,757	108,779
2078*	43,802*	65,824*	87,846*	109,868*
	43,923	65,945	87,967	109,989
22,022				
23,232	44,044	66,066	88,088	110,110
24,442	45,133*	67,155*	89,177*	111,199*
25,652	45,254	67,276	89,298	111,320
26,862	46,343*	68,365*	90,387*	112,409*
28,072	46,464	68,486	90,508	112,530
29,282	47,553*	69,575*	91,597*	113,619*
30,492	47,674	69,696	91,718	113,740
31,702	48,763*	70,785*	92,807*	114,829*
32,912	48,884	70,906	92,928	114,950
	49,973*	71,995*	94,017*	116,039*
	50,094	72,116	94,138	116,160
	51,183*	73,205*	95,227*	117,249*
	51,304	73,336	95,348	117,370
	52,393*	74,415*	96,437*	118,459*
	52,514	74,536	96,558	118,580
	53,603*	75,625*	97,647*	119,669*
	53,724	75,746	97,768	119,790
	54,813*	76,835*	98,857*	120,879*
	54,934	76,956	98,978	

Table III continued

121,000*	143,022*	165,044*	187,066*	209,088*
121,121	143,143	165,165	187,187	210,298*
122,210*	144,232*	166,254*	188,276*	211,508*
122,331	144,353	166,375	188,397	212,718*
123,420*	145,442*	167,464*	189,486*	213,928*
123,541	145,563	167,585	189,607*	215,138*
124,630*	146,652*	168,674*	190,696*	216,348*
124,751	146,773	168,795	190,817*	217,558*
125,840*	147,862*	169,884*	191,906*	218,768*
125,961	147,983	170,005*	192,027*	219,978*
127,000*	149,072*	171,094*	193,116*	
127,171	149,193	171,215*	193,437*	
128,260*	150,282*	172,304*	194,326*	
128,381	150,403*	172,425*	194,457*	
129,470*	151,492*	173,514*	195,536*	
129,591	151,613*	173,635*	195,657*	
130,680*	152,702*	174,724*	196,746*	
130,801*	152,823*	174,845*	196,867*	
131,890*	153,912*	175,934*	197,956*	
132,011*	154,033*	176,005*	198,077*	
132,132	154,154	176,176	198,198	
133,211*	155,243*	177,265*	199,287*	
133,342	155,364	177,386	199,408*	
134,431*	156,453*	178,475*	200,497*	
134,552	156,574	178,596	200,618*	
135,641*	157,663*	179,686*	201,707*	
135,762	157,784	179,806*	201,828*	
136,851*	158,873*	180,895*	202,917*	
136,972	158,994	181,016*	203,038*	
138,061*	160,083*	182,105*	204,127*	
138,182	160,204*	182,226*	204,248*	
139,272*	161,293*	183,315*	205,337*	
139,392	161,414*	183,436*	205,458*	
140,481*	162,503*	184,525*	206,547*	
140,602*	162,624*	184,646*	206,668*	
141,691*	163,713*	185,735*	207,757*	
141,812*	163,834*	185,856*	207,878*	
142,901*	164,923*	186,945*	208,967*	

Table IV

The entries of Table IV are the result of multiplying the entries of Table I by 11. An asterisk (*) indicates that the entry cannot be written in the form S=M + $\overline{M}$.

22	5544	8877	12,210*	15,543*	18,876*
44	5555	8888	12,221	15,554	18,887
66	5764	9097*	12,430*	15,783*	19,096*
88	5775	9108	12,441	15,774	19,107*
110	5984	9317*	12,650*	15,983*	19,316*
132	5995	9328	12,661	15,994	19,327*
154	6204*	9537*	12,870*	16,203*	19,536*
176	6215	9548	12,881	16,214*	19,547*
198	6424*	9757*	13,090*	16,423*	19,756*
	6435	9768	13,101*	16,434*	19,767*
2222					
2442	6655	9988	13,211*	16,654*	19,987*
2662	6666	9999	13,332	16,665	19,998
2882	6875	10,208*	13,541*	16,874*	20,207*
3102	6886	10,219	13,552	16,885	20,218*
	7095*	10,428*	13,761*	17,094*	20,427*
3322	7106	10,439	13,772	17,105*	20,438*
3333	7315*	10,648*	13,981*	17,314*	20,647*
3542	7326	10,659	13,992	17,325*	20,658*
3553	7535*	10,868*	14,201*	17,534*	20,867*
3762	7546	10,879	14,212*	17,545*	20,878*
3773					
3982	7766	11,099*	14,432*	17,765*	21,098*
3993	7777	11,110	14,443	17,776	21,318*
4202*	7986	11,319*	14,652*	17,985*	21,538*
4213	7997	11,330	14,663	17,996	21,758*
	8206*	11,539*	14,872*	18,205*	21,978*
4433	8217	11,550	14,883	18,216*	
4444	8426*	11,759*	15,092*	18,425*	
4653	8437	11,770	15,103*	18,436*	
4664	8646*	11,979*	15,312*	18,645*	
4873	8657	11,990	15,323*	18,656*	
4884					
5093*					
5313*					
5324					

<u>Theorem 2.6.</u> If $0 \le f_i + f_{i-1} \le 18$ for $1 \le i \le x-1$, $2 \le f_0 \le 18$ and $0 \le 2f_x + f_{x-1} \le 18$, then

$$11 \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1) = \sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1} + 1).$$

<u>Proof.</u>--Using the arguments of Theorem 2.5, we have

$$11 \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$$

$$= \sum_{i=0}^{x} f_i (10^{2x-i+1} + 10^{i+1}) + \sum_{i=0}^{x} f_i (10^{2x-i} + 10^i)$$

$$= \sum_{i=1}^{x} f_i 10^{2x-i+1} + \sum_{i=1}^{x} f_{i-1} 10^i + \sum_{i=1}^{x} f_{i-1} 10^{2x-i+1} + \sum_{i=1}^{x} f_i 10^i$$

$$+ f_0 10^{2x+1} + f_x 10^{x+1} + f_x 10^{x+1} + f_0$$

$$= \sum_{i=1}^{x} (f_i + f_{i-1}) 10^i (10^{2(x-i)+1} + 1) + f_0 (10^{2x+1} + 1)$$

$$+ f_x 10^x (10 + 1)$$

$$= \sum_{i=1}^{x-1} (f_i + f_{i-1}) 10^i (10^{2(x-i)+1} + 1) + f_0 (10^{2x+1} + 1)$$

$$+ (2f_x + f_{x-1}) 10^x (10 + 1)$$

$$= \sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1} + 1)$$

where $e_i = f_i + f_{i-1}$ for $1 \le i \le x-1$, $e_0 = f_0$, and $e_x = 2f_x + f_{x-1}$. Obviously, the e_i s satisfy the conditions of Part (a) of Theorem 2.1.

Table IV illustrates the results of Theorem 2.6.

At this point, one might ask if there exist numbers, other than 11, which transform classes of numbers form Table I to those of Table II and conversely. The next two theorems answer the question in the affirmative.

Theorem 2.7. Let $S = \sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1} + 1)$ and $R = 10^j + 1$ where j is odd, say $j = 2h + 1$. If $2 \le e_0 \le 18$, $0 \le e_i \le 18$ for $1 \le i \le j-1$, $0 \le e_i + e_{i-j} \le 18$ for $j \le i \le x$, $0 \le e_{i-j} + e_{2x+1-i} \le 18$ for $x+1 \le i \le h+x$, and $0 \le e_{x-h} \le 9$; then RS is a member of Table I.

Proof.--The theorem is obviously true if $j > 2k$; we assume $j \le 2k$. Following the arguments given in Theorem 2.5, we have

$$RS = \sum_{i=0}^{x} e_i 10^{2x-i+j+1} + \sum_{i=0}^{x} e_i 10^{i+j} + \sum_{i=0}^{x} e_i 10^{2x-i+1} + \sum_{i=0}^{x} e_i 10^i$$

$$= \sum_{i=0}^{x} e_i 10^{2x-i+j+1} + \sum_{i=j}^{j+x} e_{i-j} 10^i + \sum_{i=j}^{j+x} e_{i-j} 10^{2x-i+j+1} + \sum_{i=0}^{x} e_i 10^i$$

$$= \sum_{i=0}^{j-1} e_i 10^i (10^{2(x-i)+j+1} + 1) + \sum_{i=j}^{x} (e_i + e_{i-j}) 10^i (10^{2(x-i)+j+1} + 1)$$

$$+ \sum_{i=x+1}^{j+x} e_{i-j} 10^i (10^{2(x-i)+j+1} + 1).$$

Replacing j by 2h + 1 in the last sum, we have

$$\sum_{i=x+1}^{2h+x+1} e_{i-2h-1} (10^{2(x+h+1)-i} + 10^i)$$

$$= \sum_{i=x+1}^{h+x} e_{i-2h-1} 10^i (10^{2(x+h+1-i)} + 1)$$

$$+ \sum_{i=h+x+2}^{2h+x+1} e_{i-2h-1} (10^{2(h+x+1)-i} + 10^i) + 2e_{x-h} 10^{h+x+1}.$$

Recognizing that

$$\sum_{i=h+x+2}^{2h+x+1} e_{i-2h-1} (10^{2(x+h+1)-i} + 10^i) = \sum_{i=x+1}^{h+x} e_{2x+1-i} (10^{2(h+x+1)-i} + 10^i),$$

and combining the previous results, we see that

$$RS = \sum_{i=0}^{2h} e_i 10^i (10^{2(x+h+1-i)} + 1)$$

$$+ \sum_{i=2h+1}^{x} (e_i + e_{i-2h-1}) 10^i (10^{2(x+h+1-i)} + 1)$$

$$+ \sum_{i=x+1}^{h+x} (e_{i-2h-1} + e_{2x+1-i}) 10^i (10^{2(h+x+1-i)} + 1) + 2e_{x-h} 10^{x+h+1}.$$

Letting $f_i = e_i$ for $0 \leq i \leq 2h$, $f_i = (e_i + e_{i-2h-1})$ for $2h+1 \leq i \leq x$,

$f_i=(e_{i-2h-1}+e_{2x+1-i})$ for $x+1\le i\le h+x$, and $f_{h+x+1}=e_{x-h}$, we have

$RS=\sum_{i=0}^{h+x+1} f_i 10^i(10^{2(h+x+1-i)}+1)$ and the theorem is proved.

Theorem 2.8. Let $S=\sum_{i=0}^{x} f_i 10^i(10^{2(x-i)}+1)$ and $R=10^j+1$ where j is odd, say $j=2h+1$. If $2\le f_0\le 18$, $0\le f_i\le 18$ for $1\le i\le j-1$, $0\le f_i+f_{i-j}\le 18$ for $j\le i\le x-1$, $0\le 2f_x+f_{x-j}\le 18$, and $0\le f_{i-j}+f_{2x-i}\le 18$ for $x+1\le i\le h+x$ then RS is a member of Table II.

Proof.--Since the proof of Theorem 2.8 is similar to the proofs of previous theorems, the details have been omitted.

There exist other classes of numbers which will transform numbers from Table I to Table II and conversely. However, the number of restrictions becomes so large and so detailed that the author feels they are insignificant and should be left out of this paper.

In developing Theorems 2.7 and 2.8, the author found the next two results. The details of the proofs are omitted since they follow the pattern of the proof of Theorem 2.7.

<u>Theorem 2.9.</u> Let $S=\sum_{i=0}^{x} e_i 10^i(10^{2(x-i)+1}+1)$ and $R=10^j+1$ where j is even, say j=2h. If $2\le e_0\le 18$, $0\le e_i\le 18$ for $1\le i\le j-1$, $0\le e_i+e_{i-j}\le 18$ for $j\le i\le x$, and $0\le e_{i-j}+e_{2x+1-i}\le 18$ for $x+1\le i\le h+x$ then RS is of the same form and hence is in Table II.

<u>Theorem 2.10.</u> Let $S=\sum_{i=0}^{x} f_i 10^i(10^{2(x-i)}+1)$ and $R=10^j+1$ where j is even, say j=2h. If $2\le f_0\le 18$, $0\le f_i\le 18$ for $1\le i\le j-1$, $0\le f_i+f_{i-j}\le 18$ for $j\le i\le x-1$, $0\le 2f_x+f_{x-j}\le 18$, $0\le f_{i-j}+f_{2x-i}\le 18$ for $x+1\le i\le h+x-1$ and $0\le f_{x-h}\le 9$ then RS is of the same form and hence is in Table I.

In Theorems 2.3 and 2.4, it is shown that it is possible for S to be equal to $M+\overline{M}$ where $M=\overline{M}$. That is, S=2M so that 2 divides S. We shall now show that this is not possible for any integer larger than two.

<u>Theorem 2.11.</u> If $S=M+\overline{M}$ then there does not exist any integer $q>2$ such that $qM=S$.

<u>Proof.</u>--We wish to show that there does not exist an integer $q>2$ such that $qM=M+\overline{M}$. To do this, we shall assume that such an integer does exist and arrive at a contradiction.

If $qM=M+\bar{M}$ where $q>2$, then $(q-1)M=\bar{M}$ where $(q-1)>1$. Recalling that $M=\sum_{i=0}^{k} a_i 10^i$ and $\bar{M}=\sum_{i=0}^{k} a_{k-i} 10^i$ and comparing like powers of ten, we obtain the equations $(q-1)a_0=a_k$ and $(q-1)a_k=a_0$. Hence,

$$(q-1)^2 a_k=a_k$$

so that $a_k=0$ or $(q-1)=\pm 1$. However, $a_k\neq 0$ and $(q-1)>1$ so that our assumption is false and the theorem is proved.

The last two theorems of this section deal with the number of ways in which we can write $S=M+\bar{M}$.

<u>Theorem 2.12.</u> Let $S=M+\bar{M}=\sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1}+1)$ where $e_i=a_i+a_{2x+1-i}$ and is such that $0\leq e_i\leq 18$ for $1\leq i\leq x$ and $2\leq e_0\leq 18$. Then there exist

$$(9-|10-e_0|)\prod_{i=1}^{x}(10-|9-e_i|) \text{ possible values of } M.$$

<u>Proof.</u>--We shall use the arguments found on page 9.

First we observe that a_0 must lie in the range stated in (2.5) and that a_i for $i\leq i\leq x$ must lie in the range stated in (2.7).

By (2.5), we see that $0<a_0<e_0$ if $e_0\leq 10$ and $e_0-10<a_0<10$ if $e_0>10$. In the former case, there exist $e_0-1=9-|10-e_0|$ choices for a_0

while there are $19-e_0=9-|10-e_0|$ choices if $e_0>10$. In either case, there are $9-|10-e_0|$ choices of a_0.

By (2.7), we see that $0\leq a_i\leq e_i$ if $e_i\leq 9$ and $e_i-9\leq a_i\leq 9$ if $e_i>9$. Hence, there exist $e_i + 1=10-|9-e_i|$ choices for a_i if $e_i\leq 9$ and there exist $19-e_i=10-|9-e_i|$ choices for a_i if $e_i>9$. In either case, there are $10-|9-e_i|$ choices for a_i.

Combining (2.9) with the conclusions of the two preceding paragraphs, we see that there exist

$$(9-|10-e_0|)\prod_{i=0}^{x}(10-|9-e_i|)$$

possibilities for M where $\prod_{i=1}^{x} v_i$ is the product of the integers $v_1, v_2, v_3, \ldots, v_x$.

Using (2.10) through (2.14) and an argument similar to that of the preceding theorem, we have

<u>Theorem 2.13.</u> If $S=M + \bar{M}= \sum_{i=0}^{x} f_i 10^i(10^{2(x-i)} + 1)$ where $f_i=a_i + a_{2x-i}$ and is such that $0\leq f_i\leq 18$ for $1\leq i\leq x-1$, $0\leq f_x\leq 9$, and $2\leq f_0\leq 18$ then there exist

$$(9-|10-f_0|)\prod_{i=1}^{x-1}(10-|9-f_i|)$$ possible values of M.

3. A Partial Solution

At this point, we have established a generator for all numbers of the form $S=M + \bar{M}$ and have made several statements concerning the nature of this set of numbers. We know there are infinitely many numbers of the form $M + \bar{M}$ since there exist infinitely many replacements for M and given any $n \geq 1$, we suspect that there are infinitely many which are divisible by n.

We can see from an examination of Table V that for any $n \leq 100$, there does exist at least one multiple of n which is also the sum of reversed digit addends. A further interesting note on the numbers of Table V is that if $n \equiv 0 \pmod{10}$ then the first S which is also a multiple of n is frequently a much larger number than that required for neighboring n's not divisible by 10. This is so because the last digit of S is zero. This also gives us a clue as to how one might seek a proof of the initial question.

Throughout this chapter, we will restrict our consideration to numbers of the form $M + \bar{M} = \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$ keeping in mind that parallel statements could be made for the case $\sum_{i=0}^{x} e_i 10^i (10^{2(x-i)+1} + 1)$.

We will first show that if $(10,n)=1$ then there exists an $M + \bar{M} = \sum_{i=0}^{x} f_i 10^i (10^{2(x-i)} + 1)$ which is divisible by n. Secondly, we shall show that there are in fact infinitely many solutions provided $(10,n)=1$. Finally, we shall discuss the case for $(10,n)>1$.

Table V

Given an $n \leq 100$, there exists an $S = M + \overline{M}$ such that $S \equiv 0 \pmod{n}$.

n	S	n	S	n	S
2	2	41	11,271	81	1,999,999,998
3	6	42	504	82	8118
4	4	43	1333	83	747
5	10	44	44	84	504
6	6	45	585	85	50,405
7	14	46	322	86	23,822
8	8	47	282	87	21,402
9	18	48	1392	88	88
10	10	49	343	89	5874
11	22	50	1050	90	10,890
12	12	51	969	91	2002
13	132	52	988	92	828
14	14	53	424	93	5115
15	165	54	1998	94	282
16	16	55	55	95	665
17	187	56	504	96	23,232
18	18	57	342	97	24,832
19	323	58	464	98	686
20	10,120	59	767	99	99
21	483	60	10,560	100	11,000
22	22	61	16,775		
23	828	62	868		
24	504	63	504		
25	525	64	2112		
26	988	65	585		
27	1998	66	66		
28	504	67	8107		
29	464	68	544		
30	1050	69	483		
31	403	70	10,010		
32	544	71	34,932		
33	33	72	504		
34	544	73	584		
35	525	74	222		
36	504	75	525		
37	444	76	988		
38	342	77	77		
39	585	78	20,202		
40	10,120	79	948		
		80	10,560		

<u>Theorem 3.1.</u> If $(10,n)=1$ then there exists an $S=M+\overline{M}\equiv 0 \pmod{n}$ such that $M+\overline{M}=\sum_{i=0}^{x} f_i 10^i(10^{2(x-i)}+1)$.

<u>Proof.</u>--Let $S=\sum_{i=0}^{n-1} f_i 10^i(10^{2(n-1-i)}+1)$ where $2\le f_0\le 18$ and $2\le f_i\le 9$ for $1\le i\le n-1$.

Using Theorem 1.1 and Definition 1.1, we obtain the following congruences:

(3.1) $$f_0(10^{2(n-1)}+1)=q_0 n+r_0\equiv r_0 \pmod{n}$$

$$f_1 10(10^{2(n-2)}+1)+r_0=q_1 n+r_1\equiv r_1 \pmod{n}$$

$$f_2 10^2(10^{2(n-3)}+1)+r_1=q_2 n+r_2\equiv r_2 \pmod{n}$$

...

$$f_{n-1}10^{n-1}(10^0+1)+r_{n-2}=q_{n-1}n+r_{n-1}\equiv r_{n-1} \pmod{n}.$$

If no two of the remainders are equal in the above system of n congruences then there is a j such that $r_j\equiv 0 \pmod{n}$. Hence,

(3.2) $$f_j 10^j(10^{2(n-1-j)}+1)+r_{j-1}\equiv 0 \pmod{n}.$$

Let $S'=\sum_{i=0}^{n-1} f'_i 10^i(10^{2(n-1-i)}+1)$ where

(3.3) $$f'_i=\begin{cases} f_i, & 1\le i\le j \\ 0, & j+1\le i\le n-1. \end{cases}$$

By Theorem 2.1, it is obvious that S' is of the form $M + \overline{M}$.

Furthermore,

(3.4) $$S' = \sum_{i=0}^{j} f_i 10^i (10^{2(n-1-i)} + 1).$$

From (3.1), we see that

(3.5) $$\sum_{i=0}^{j} f_i 10^i (10^{2(n-1-i)} + 1) + \sum_{i=0}^{j-1} r_i \equiv \sum_{i=0}^{j} r_i \pmod{n}.$$

Combining (3.4) and (3.5), we have

(3.6) $$S' \equiv r_j \equiv 0 \pmod{n},$$

and we are done if all the remainders are distinct.

Suppose that in (3.1), we have $r_b = r_c$ where $s \leq b < c \leq n-1$. Then

(3.7) $$\sum_{i=0}^{b} f_i 10^i (10^{2(n-1-i)} + 1) \equiv \sum_{i=0}^{c} f_i 10^i (10^{2(n-1-i)} + 1) \pmod{n}.$$

Hence,

(3.8) $$\sum_{i=b+1}^{c} f_i 10^i (10^{2(n-1-i)} + 1) = 10^{b+1} \sum_{i=b+1}^{c} f_i 10^i (10^{2(n-1-i)} + 1)$$

$$= 10^{b+1} \sum_{i=0}^{c-b-1} f_{b+1+i} 10^i (10^{2(n-b-2-i)} + 1)$$

$$\equiv 0 \pmod{n}.$$

However, $(n,10)=1$ so that $(n,10^{b+1})=1$. Therefore, (3.8) becomes

(3.9) $$\sum_{i=0}^{c-b-1} f_{b+1+i} 10^i (10^{2(n-b-2-i)} + 1) \equiv 0 \pmod{n}.$$

Let $S' = \sum_{i=0}^{n-b-2} f_i' 10^i (10^{2(n-b-2-i)} + 1)$ where

$$f_i' = \begin{cases} f_{b+1+i}, & 0 \le i \le c-b-1 \\ 0, & c-b \le i \le n-b-2 \end{cases} \tag{3.10}$$

Since S' satisfies the hypotheses of Theorem 2.1, we know that $S' = M + \overline{M}$. Furthermore, it is obvious by (3.9) and (3.10) that $S' \equiv 0 \pmod{n}$ and we are done.

Recalling that Theorem 2.10 is obviously true if $j > 2x$, we have

Theorem 3.2. If $(n,10)=1$ then there exist infinitely many $M + \overline{M} \equiv 0 \pmod{n}$.

Before considering the case where $(n,10)>1$, we note the following.

Theorem 3.3. If $(n,10)=1$ then there are at least 17 multiples of n that are sums of reversed digit addends in the range $2 \le S \le 2 \cdot 10^{2n-1}$.

Proof.--Replacing f_0 in (3.1) by any value between 2 and 18 other than the initially specified value of f_0 will merely change each r_i to $r_i + K$ where K is some constant between 0 and n-1. Hence, one of the $r_i + K$ is congruent to zero modulo n if the r_i were distinct or $r_b + K \equiv r_c + K \pmod{n}$. In either case, the argument of Theorem 3.1 gives a new $M + \overline{M}$. Since S is obviously between 2 and $2 \cdot 10^{n-1}$, the theorem is proved.

Although we cannot answer the question completely when $(10,n)>1$, we can find a solution for many special cases. Using Theorems 2.9 and 2.10, we can then construct infinitely many solutions to $M + \overline{M} \equiv 0 \pmod{n}$.

Theorem 3.4. Let $(10,d)=1$. If there exists an integer t such that $10^t \equiv -1 \pmod{d}$ then 10d divides a number of the form $M + \overline{M}$.

Proof.--Since $10^t \equiv -1 \pmod{d}$, we have $10(10^t + 1) \equiv 0 \pmod{10d}$. Obviously, $10(10^t + 1)$ is a sum of reverse digit addends.

Theorem 3.5. Let $n=33 \cdot 10^s$. Let $S=\sum_{i=0}^{2s+1} f_i 10^i (10^{2(2s+1-i)} + 1)$ where $f_0=10$, $f_i=9$ for $1 \le i \le 2s-2$, $f_{2s-1}=f_{2s+1}=2$, and $f_{2s}=16$. Then $S \equiv 0 \pmod{33 \cdot 10^s}$ or $33 \cdot 10^s$ divides a sum of reverse digit addends.

Proof.--Expanding S, we see that

(3.11) $$S=\sum_{i=0}^{2s-2} f_i 10^i + \sum_{i=0}^{2s-2} f_i 10^{4s+2-i} + 20002\cdot 10^{2s-1} + 1616\cdot 10^{2s} + 40\cdot 10^{2s+1}$$

$$\equiv \sum_{i=0}^{2s-2} f_i 10^i \pmod{10^s}.$$

Now,

(3.12) $$\sum_{i=0}^{2s-2} f_i 10^i = 10 + 9(10 + 10^2 + \ldots + 10^{2s-2})$$

$$=1 + 9(1 + 10 + 10^2 + \ldots + 10^{2s-2})$$

$$=1 + (10-1)(1 + 10 + 10^2 + \ldots + 10^{2s-2})$$

$$=10^{2s-1}.$$

Hence, we have $S\equiv 0 \pmod{10^s}$.

Furthermore, noting that $10^{2q}\equiv 1 \pmod{33}$ for any q, we obtain

(3.13) $$S\equiv 2\sum_{i=0}^{2s+1} f_i 10^i \pmod{33}$$

$$\equiv 2[\sum_{i=0}^{2s-2} f_i 10^i + 20 + 16 + 20] \pmod{33}$$

$$\equiv 2(66) \pmod{33}$$

$$\equiv 0 \pmod{33}.$$

Since $(33,10)=1$, we have $(33,10^s)=1$ so that $S\equiv 0 \pmod{33\cdot 10^s}$.

The next theorem can be proved in a manner similar to that of Theorem 3.5. Hence, the details are omitted.

Theorem 3.6. (a) If $n=3\cdot 10^s$ let $S=\sum_{i=0}^{2s+1} f_i 10^i(10^{2(2s+1-i)}+1)$ where $f_0=10$, $f_i=9$, for $1\le i\le 2s-2$, $f_{2s-1}=f_{2s}=0$, and $f_{2s+1}=2$. Then $S\equiv 0 \pmod{3\cdot 10^s}$.

(b) If $n=9\cdot 10^s$ let $S=\sum_{i=0}^{2s+1} f_i 10^i(10^{2(2s+1-i)}+1)$ where $f_0=10$, $f_i=9$ for $1\le i\le 2s-2$, $f_{2s-1}=f_{2s}=0$, and $f_{2s+1}=8$. Then $S\equiv 0 \pmod{9\cdot 10^s}$.

In conclusion, we observe the following.

Let $n=2^p5^qd$ where $(10,d)=1$. Let $s=\text{maximum}\{p,q\}$ so that $n|10^sd$. Define $S=\sum_{i=0}^{d+s-1} f_i 10^i(10^{2(d+s-1-i)}+1)$ where $f_0=10$, $f_i=9$ for $1\le i\le s-1$, and $0\le f_i\le 9$ for $s\le i\le d+s-1$.

Following the argument of (3.11) and (3.12), we see that $S\equiv 0 \pmod{10^s}$. If it is also true that $S\equiv 0 \pmod{d}$ we are done. If not, we proceed as in the proof of Theorem 3.1 and we write

(3.14) $$U + f_s 10^s(10^{2(d-1)}+1)\equiv r_s \pmod{d}$$

$$r_s + f_{s+1}10^{s+1}(10^{2(d-2)}+1)\equiv r_{s+1} \pmod{d}$$

$$r_{s+1} + f_{s+2}10^{s+2}(10^{2(d-3)} + 1) \equiv r_{s+2} \pmod d$$

$$\cdots$$

$$r_{d+s-2} + 2f_{d+s-1}10^{d+s-1} \equiv r_{d+s-1} \pmod d$$

where $U = \sum_{i=0}^{s-1} f_i 10^i (10^{2(d+s-1-i)} + 1)$.

If there exists a j such that r_j is congruent to zero modulo d, then we can follow the argument of Theorem 3.1 to show that

$$U + \sum_{i=s}^{j} f_i 10^i (10^{2(d+s-1-i)} + 1) \equiv 0 \pmod d. \tag{3.15}$$

Let

$$S' = \sum_{i=0}^{d+s-1} f_i' 10^i (10^{2(d+s-1-i)} + 1) \text{ where} \tag{3.16}$$

$$f_i' = \begin{cases} 10, & i=0 \\ 9, & 0 \le i \le s-1 \\ f_i, & s \le i \le j \\ 0, & j+1 \le i \le d+s-1. \end{cases}$$

It is now obvious that S' is of the form $M + \overline{M}$, $S' \equiv 0 \pmod d$, and $S' \equiv 0 \pmod{10^s}$. However, $(10,d)=1$ so that $(10^s,d)=1$. Hence, $S' \equiv 0 \pmod{10^s d}$ or $S' \equiv 0 \pmod n$.

If there does not exist an r_j in (3.14) which is congruent to zero modulo d, we construct the following set of congruences:

(3.17)
$$f_s 10^s(10^{2(d-1)} + 1) \equiv q_s \pmod d$$

$$q_s + U + f_{s+1}10^{s+1}(10^{2(d-2)} + 1) \equiv r_{s+1} \pmod d$$

$$r_{s+1} + f_{s+2}10^{s+2}(10^{2(d-3)} + 1) \equiv r_{s+2} \pmod d$$

$$\ldots$$

$$r_{d+s-2} + 2f_{d+s-1}(10^{d+s-1}) \equiv r_{d+s-1} \pmod d.$$

Since there does not exist an $r_j \equiv 0 \pmod d$, we ask if there does exist an $r_j \equiv q_s \pmod d$ where $s+1 \le j \le d+s-1$. If so, then by using the arguments of (3.7) and (3.8) in Theorem 3.1, we have

(3.18)
$$U + \sum_{i=s+1}^{j} f_i 10^i(10^{2(d+s-1-i)} + 1) \equiv 0 \pmod d.$$

Let

(3.19)
$$S' = \sum_{i=0}^{d+s-1} f_i' 10^i(10^{2(d+s-1-i)} + 1) \text{ where}$$

$$f_i' = \begin{cases} 10, & i=0 \\ 9, & 1 \le i \le s-1 \\ 0, & i=s \\ f_i, & s+1 \le i \le j \\ 0, & j+1 \le i \le d+s-1. \end{cases}$$

It is obvious that S' is of the form $M + \overline{M}$ and that $S' \equiv 0 \pmod{10^s d}$. Since it is possible that $q_s \not\equiv r_j \pmod d$ for $s+1 \le j \le d+s-1$, we write

(3.20)
$$f_s 10^s(10^{2(d-1)} + 1) \equiv q_s (\text{mod } d)$$

$$q_s + f_{s+1}10^{s+1}(10^{2(d-2)} + 1) \equiv q_{s+1}(\text{mod } d)$$

$$q_{s+1} + U + f_{s+2}10^{s+2}(10^{2(d-3)} + 1) \equiv r_{s+2}(\text{mod } d)$$

$$\cdots$$

$$r_{d+s-2} + 2f_{d+s-1}10^{d+s-1} \equiv r_{d+s-1}(\text{mod } d).$$

As before, we ask if there exists an r_j for $s+2 \le j \le d+s-1$ such that $q_{s+1} \equiv r_j (\text{mod } d)$. If there is, we proceed as before except that we let

(3.21)
$$f'_i = \begin{cases} 10, & i=0 \\ 9, & 1 \le i \le s-1 \\ 0, & i=s \\ 0, & i=s+1 \\ f_i, & s+2 \le i \le j \\ 0, & j+1 \le i \le d+s-1. \end{cases}$$

If a suitable r_j does not exist, we repeat the preceding process, placing U in successive congruences until, with U in the (g+1)st congruence, we might find an $r_j \equiv q_{g-1}(\text{mod } d)$ where $g \le j \le d+s-1$. We then have

(3.22)
$$U + \sum_{i=g}^{j} f_i 10^i(10^{2(d+s-1-i)} + 1) \equiv 0(\text{mod } d).$$

Let

$$S' = \sum_{i=0}^{d+s-1} f'_i 10^i(10^{2(d+s-1-i)} + 1) \text{ where}$$

$$(3.23) \qquad f_i' = \begin{cases} 10, & i=0 \\ 9, & 1\le i\le s-1 \\ 0, & s\le i\le g-1 \\ f_i, & g\le i\le j \\ 0, & j+1\le i\le d+s-1. \end{cases}$$

If having added U to every congruence in the manner described in (3.12), (3.17) and (3.20) up to the point where we have

$$U + q_{d+s-2} + 2f_{d+s-1}10^{d+s-1} \equiv r_{d+s-1} \pmod d$$

does not give us $q_{d+s-2} \equiv r_{d+s-1} \pmod d$, we must conclude that the procedure described in this chapter will fail to construct a number S which is of the form $M + \overline{M}$ and is also divisible by n. We believe that if we vary the values of f_i for $s\le i\le d+s-1$ and repeat the procedure outlined in (3.17) through (3.23) we will arrive at the desired solution. The proof, however, escapes discovery.

REFERENCES

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