Mathematical Simulation of Sediment Deposit in Estuaries

Yuh-Hua Tsai

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MATHEMATICAL SIMULATION OF
SEDIMENT DEPOSIT IN ESTUARIES

BY

YUH-HUA TSAI

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Civil Engineering, South Dakota
State University

1972
MATHEMATICAL SIMULATION OF
SEDIMENT DEPOSIT IN ESTUARIES

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser    Date

Head, Civil Engineering    Date
Départment
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I. INTRODUCTION

To predict sediment deposit in tidal estuaries is very difficult partly because of the complex physical phenomena in river estuaries, and partly because of the fact that no satisfactory mathematical method for such analysis is available. However, because of the remarkable improvement in electronic digital computers it is now possible, by reasonable approximation, to formulate a mathematical model in sufficient detail to permit the computation of sediment deposition in estuaries.

In previous investigations (1,2), the analyses were restricted to channels with constant width, and only simplified solutions have been obtained. In order to compute transient flows in long reaches of rivers or estuaries having variable cross sections, Lai (1) has developed a computer program, whereby the overall reach was divided into many subreaches and each one was considered to have uniform geometry. However, only a channel with rigid boundary was treated in his study. Three basic partial differential equations which describe the two co-existing and mutually interfering phenomena between the flow unsteadiness and the bed changing process have been derived by Hsu and Chu (2,3,4) for channels of constant width. Four additional systems of characteristic equations and their compatibility equations have also been derived depending on various assumptions. Owing to the difficulty of carrying out the calculations, only graphical solutions were
obtained.

It is viewed as necessary and useful to devise a method which can be used to solve the problems of sediment deposit in estuaries more directly and completely. In this study, three basic equations have been extended to a more general case in which lateral inflows and various width of channel are considered.

A solution of the problem has been obtained by Chang (3, 4) and Richards (4). In their research, three basic equations have been successfully reduced to two differential equations by assuming that the change in bed slope due to sediment deposit in a short period is insignificant compared with the original bed slope. Based upon these two differential equations, the method of characteristics has been used to solve the problem, and the computing procedures are divided into two parts: first, to compute the average velocity and depth of sediment-laden water, second, to estimate the deposition of sediment by using the continuity equation of sediment.

In the following chapters, a solution of these three nonlinear differential equations is presented with mathematical analyses, computer approach and some notes in programming. The method of characteristics has been employed in this study. The analysis is based on the assumptions that the fluid has homogeneous density, the flow has uniform velocity in any cross section, and uniform diameters for sediment particles.
It is also assumed that the bedform change due to the effect of bedload is neglected. A characteristic equation in the form of third order polynomial can be derived, and solved by the numerical solution (5) instead of graphical solution.

The specific purposes of this research are to investigate the change of riverbed and water surface configurations due to sediment deposition in river estuaries, to extend the three basic equations for a more general case, and to establish a computer program for solving these equations by using the method of characteristics.
II. THEORETICAL DEVELOPMENT

Fundamental equations that describe river configurations of one-dimensional open-channel transient flow are the equations of the conservation of mass and momentum. The analysis is based on the assumptions that the flow in the channel is of substantially homogeneous density, that the velocity is uniform over any cross section, and that hydrostatic pressure prevails at any point in the channel. The channel is assumed to be sufficiently straight and rectangular in shape throughout the reach. Under these assumptions, three basic partial differential equations for solving transient flows along a reach of a river can be derived as follows:

\[
\frac{\partial}{\partial x}(vbhc) + \frac{\partial}{\partial t}(bhc) + \rho \frac{\partial}{\partial t}(bz) - Q_S = 0 \tag{1}
\]

\[
\frac{\partial}{\partial x}(vbh) + \frac{\partial}{\partial t}(bh) + \frac{\partial}{\partial t}(bz) - Q_m = 0 \tag{2}
\]

\[
\frac{\partial}{\partial t}(\rho_mvbh) + \frac{\partial}{\partial x}(\rho_m v^2 bh) = -\frac{\partial}{\partial x}(\rho_m g bh^2) - \rho_m g bh \left( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \right) - b \tau_o \tag{3}
\]

Equation 1 is the equation of continuity for sediment, Equation 2 is the equation of continuity for sediment-laden water, and Equation 3 is the equation of motion for sediment-laden water. In these three equations, the flow depth, \( h \), the flow velocity, \( v \), and the thickness of sediment, \( z \), are used
as three dependent variables. The distance measured in the flow direction, \( x \), and the time, \( t \), are used as two independent variables. The variable coefficient, \( c \), represents the volume of sediment in a unit volume of sediment-laden water (concentration), while \( b \) is the top width of the channel cross section; \( p \) is the volume of solids in unit volume of bed sediment; \( Q_s \) and \( Q_m \) are lateral inflow of sediment and sediment-laden water for unit length of channel; \( Z \) is the elevation of the fixed riverbed, and \( g \), \( \tau_0 \), and \( \rho_m \) represent gravitational acceleration, boundary shear, and the density of the sediment-laden water, respectively. The definition sketch is shown in Figure 1.

![Definition Sketch of an Alluvial Channel](image-url)
In solving the flows based upon the above three equations, the sediment concentration, \(c\), at any instant at any cross section is assumed to be proportional to the sediment transport capacity of the flow, and hence may be written as a function of \(v\) and \(h\), i.e.,

\[
c = \frac{K}{g_w} v^m h^n
\]

where \(w\) is the average fall velocity of sediment, \(g\) is the gravitational acceleration, and \(K\) is the coefficient of sediment transport capacity. The values of \(m\) and \(n\) can be obtained from the study of field data collected in the very river under investigation. Differentiating Equation 4 with respect to \(x\) and \(t\) gives the following equations

\[
\frac{\partial c}{\partial x} = \frac{mc}{v} \frac{\partial v}{\partial x} + \frac{nc}{h} \frac{\partial h}{\partial x}
\]

\[
\frac{\partial c}{\partial t} = \frac{mc}{v} \frac{\partial v}{\partial t} + \frac{nc}{h} \frac{\partial h}{\partial t}
\]

The density of the sediment-laden water, \(\rho_m\), can be expressed in terms of the density of water, \(\rho_w\), the density of sediment, \(\rho_s\), and the sediment concentration, \(c\), as

\[
\rho_m = c \rho_s + (1 - c) \rho_w = (\rho_s - \rho_w)c + \rho_w
\]

The boundary shear may be expressed in the following form:

\[
\tau_o = \frac{\rho_m g N^2}{h^{1/3} v|v|}
\]

where \(N\) is Manning's roughness coefficient and the absolute-
value sign indicates that $\tau_0$ is positive (in the direction from right to left) when $v$ is positive (in the direction from left to right).
III. A COMPUTER SOLUTION OF THE DIFFERENTIAL EQUATIONS
BY THE METHOD OF CHARACTERISTICS

The governing partial differential equations, presented
in the previous chapter, can be solved by the method of char-
acteristics with the aid of a computer, using specified time
intervals and an interpolation procedure, as described in the
following sections.

A. Derivation of the Characteristic Equations

By substituting Equation 4 into Equation 1, and taking
the partial derivative with respect to x and t, Equation 1
becomes

\[ \frac{K}{gw} \frac{\partial}{\partial x} (v^{m+1}h^{n+1}) + \frac{K}{gw} \frac{\partial}{\partial t} (v^{m+1}h^{n+1}) + p \frac{\partial}{\partial t} (bz) - Q_s = 0. \] (9)

Assuming \( \frac{\partial b}{\partial t} = 0 \), Equation 9 can be rearranged as

\[ J_1 = \frac{(m+1)ch}{p} v_x + \frac{mch}{pv} v_t + \frac{(n+1)cv}{p} h_x + \frac{(n+1)ch}{p} h_t \]
\[ + \frac{cvh}{pb} b_x - \frac{Q_s}{pb} = 0. \] (10)

Similarly, Equation 2 can be rewritten as

\[ J_2 = hv_x + vh_x + h_t + z_t + \frac{vh}{b} b_x - \frac{Q_m}{b} = 0 \] (11)

where \( v_x = \frac{\partial v}{\partial x} \), \( h_t = \frac{\partial h}{\partial t} \), etc..

Again, substituting Equation 4 into Equation 3, and
rearranging it, the terms on the right-hand side can be
written as
and the terms on the left-hand side of the equation can be rearranged as

\[
\frac{\partial}{\partial t}(\rho m v_{bh}) + \frac{\partial}{\partial x}(\rho m v^2_{bh}) = \rho m v \frac{\partial}{\partial t}(bh) + \frac{\partial}{\partial x}(v_{bh}) + bh \frac{\partial}{\partial t}(\rho m v) + v_{bh} \frac{\partial}{\partial x}(\rho m v)
\] (13)

From Equation 7,

\[
\frac{\partial \rho m}{\partial x} = (\rho_s - \rho_m) \frac{\partial c}{\partial x}
\] (14)

and

\[
\frac{\partial \rho m}{\partial t} = (\rho_s - \rho_m) \frac{\partial c}{\partial t}
\] (15)

In which, \( \rho_s \) and \( \rho_w \) are assumed to be constants.

Substituting Equations 14, 15 and 2 into Equation 13, the equation becomes

\[
\frac{\partial}{\partial t}(\rho m v_{bh}) + \frac{\partial}{\partial x}(\rho m v^2_{bh}) = \rho m v \left[Q_m - \frac{\partial}{\partial t}(bz) \right] + bh \rho m \frac{\partial v}{\partial t} + v_{bh} \rho m \frac{\partial v}{\partial x} + v_{bh}(\rho_s - \rho_w) \left[\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} \right]
\] (16)

From Equations 1 and 2

\[
bh \frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = Q_s - cQ_m + (c - p) \frac{\partial}{\partial t}(bz)
\]

Substituting this relationship into Equation 16, the process yields
\[ \frac{\partial}{\partial t}(\rho_m v_{bh}) + \frac{\partial}{\partial x}(\rho_m v^2_{bh}) = bh\rho_m\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) - v\left( (1 - p) \rho_w + p \rho_s \right) \frac{\partial}{\partial t}(bz) + v \rho_w Q_m + v(\rho_s - \rho_w)Q_s \]  

Equating 12 and 17, and dividing by \( \rho_m bh \), Equation 3 finally becomes

\[ J_3 = \left( v + \frac{m c g h (\rho_s - \rho_w)}{\rho_m} \right)(\partial v/\partial x) + \left( \frac{\partial v}{\partial t} \right) - v \left[ \frac{\rho_s (1-p) \rho_w}{\rho_m} \right] \left( \frac{\partial z}{\partial t} \right) 
\]

\[ + \left[ g + \frac{ncg}{2}(\rho_s - \rho_w) \right] (\partial h/\partial x) + g(\partial z/\partial x) + \frac{v}{bh} \left[ \frac{\rho_s Q_s - \rho_w (\rho_s - \rho_m)}{\rho_m} \right] + \frac{eh}{2b}(\partial b/\partial x) - g(i_o - i_f) = 0 \]  

In which, \( i_0 \) is the slope of fixed bed which may be written as \(- \frac{\partial z}{\partial x}\), and \( i_f \) replaces the term \( \frac{\tau_o}{\rho_m gh} \).

Equations 10, 11, and 18 are three simultaneous quasi-linear partial differential equations of the first order with two independent and three dependent variables. A linear combination of these equations leads to

\[ \text{Eq. 10} + \text{Eq. 11} \times \Psi_1 + \text{Eq. 18} \times \Psi_2 \]

\[ = \left( v + \frac{m}{2} A v^{m-1} h^{n+1} + (m + 1) B v^m h^n \Psi_1 + h \Psi_2 \right) \left( \frac{\partial v}{\partial x} \right) 
\]

\[ + \left[ 1 + m B v^{m-1} h^n \Psi_1 \right] \left( \frac{\partial v}{\partial t} \right) + \left[ (n + 1) B v^m h^n \Psi_1 + g \right. 
\]

\[ + \frac{n}{2} A v^m h^n + v \Psi_2 \left( \frac{\partial h}{\partial x} \right) + \left[ (n + 1) B v^m h^n + \Psi_2 \right] \left( \frac{\partial h}{\partial t} \right) + g \left( \frac{\partial z}{\partial x} \right) 
\]

\[ + \left[ \Psi_1 + \Psi_2 - \frac{C v}{h} \right] \left( \frac{\partial z}{\partial t} \right) = g(i_o - i_f) - \frac{\partial b}{\partial x} \left[ \frac{B v^m h^n + 1}{b} \right] \Psi_1 \]
\[
- \frac{v h \Psi_2}{b} = \frac{v}{b h} \left( \frac{\rho_s - \rho_w}{\rho_m} Q_s + Q_m \right)
\]  \hspace{1cm} (19)

where \( \Psi_1 \) and \( \Psi_2 \) are unknown functions of \( x \) and \( t \), and

\[
A = K \frac{\rho_s - \rho_w}{\rho_m}, \quad B = \frac{K}{g^p}, \quad \text{and} \quad C = \frac{p \rho_s + (1 - p) \rho_w}{\rho_m}.
\]

If \( v = v(x,t) \), \( h = h(x,t) \), and \( z = z(x,t) \) are solutions to Equations 10, 11, and 18, then the total differentials of \( v, h, \) and \( z \) are

\[
\frac{d}{dt}(v, h, z) = \frac{\delta}{\delta x}(v, h, z) \frac{dx}{dt} + \frac{\delta}{\delta t}(v, h, z) \hspace{1cm} (20)
\]

Now, by examination of Equation 19, with Equation 20 in mind, it can be noted that

\[
\lambda = \frac{dx}{dt} = \frac{v + \frac{m A v}{2} v_{m-1} h^{n+1} + (m+1) B v_{m} h^{n+1} \Psi_1 + h \Psi_2}{1 + m B v_{m-1} h^{n+1} \Psi_1}
\]

\[
= \frac{(n+1) B v_{m+1} h^{n} \Psi_2 + g + \frac{n A v}{2} v_{m} h^{n} + v \Psi_2}{(n+1) B v_{m} h^{n} + \Psi_2}
\]

\[
\lambda = \frac{g}{\Psi_1 + \Psi_2 - C \frac{v}{h}}, \hspace{1cm} (21)
\]

Equation 19 can be transformed into ordinary differential equations of \( v, h, \) and \( z \). From this subsidiary condition, \( \Psi_1 \) and \( \Psi_2 \) can be found as

\[
\Psi_1 = - \frac{n A v_{m} h^{n} \lambda - g v + \lambda (\lambda - v) C \frac{v}{h} + (n+1) B v_{m} h^{n} \lambda^2}{\lambda (\lambda - v) + \lambda (n+1) B v_{m+1} h^{n}} \hspace{1cm} (22)
\]

and
Substituting Equations 22 and 23 into the equality of Equation 21, which still has not been used, a characteristic equation of the third order can be obtained. That is

\[
\frac{g + \frac{n}{2} A v^m h^n + (n+1) B v^m h^n (r - \lambda + C v^2)}{(\lambda - v) + (n+1) B v^{m+1} h^n}
\]

(23)

Substituting Equations 22 and 23 into the equality of Equation 21, which still has not been used, a characteristic equation of the third order can be obtained. That is

\[
\left(1 + m B C v^m h^n + m(n+1) B^2 v^{m-1} h^n + n B v^{m-1} h^n + (n+1) B v^m h^n\right) \lambda^3 - \left\{2 v + \frac{n}{2} A v^m h^n + (n+1) B v^{m+1} h^n \right\} \lambda^2 + \left\{v^2 + \frac{1}{2} (m-n) A v^m h^n + (n+1) B v^{m+1} h^n \right\} \lambda + (m-n)
\]

\[g B v^{m+1} h^n = 0 \quad (24)\]

Three real roots \(\lambda\)'s which satisfy the above equation can be found by the numerical method described in Appendix C, and three different characteristic curves, denoted by \(C_1, C_2,\) and \(C_3,\) at the point \((x,t)\) can be found.

Substituting Equations 21, 22, and 23 into Equation 19, a total differential equation associated with each one of these three characteristic curves can be obtained.

\[
\left(1 + m B v^{m-1} h^n + n B v^{m+1} h^n \right) \left(\frac{d v}{d t}\right) + \left[(n+1) B v^m h^n + \psi_2\right] \left(\frac{d h}{d t}\right)
+ (\psi_1 + \psi_2 - c v h) \left(\frac{d z}{d t}\right) = g (i_0 - i_f) - \frac{\partial b}{\partial x} \left\{\frac{B v^{m+1} h^n}{b} \psi_1 \right\}
+ \frac{v h}{b} \psi_2 + \frac{c h}{2 b} \right) - \frac{v}{b h} \left(\frac{\rho_s Q_s - \rho_w(\rho_s - \rho_m)}{\rho_m} \right) - \frac{Q_s \psi_1}{p b} \psi_1 - \frac{Q_m \psi_2}{b} \psi_2. \quad (25)
\]
Rewriting Equation 25 as

\[ \frac{D}{dt} \frac{dv}{dt} + \frac{E}{dt} \frac{dh}{dt} + \frac{F}{dt} \frac{dz}{dt} = g(i_o - i_T) - Q - R, \]  

(26)

where

\[ D = 1 + m B v^{m-1} h^{n+1} \psi_1, \]
\[ E = (n + 1) B v^m h^n + \psi_2, \]
\[ F = \psi_1 + \psi_2 - C \frac{v}{h}, \]
\[ Q = \frac{\delta b}{\delta x} \left( B \frac{v^{m+1} h^{n+1}}{b} \psi_1 + \frac{v h}{b} \psi_2 + \frac{g h}{2b} \right), \]
\[ R = \frac{v}{b h} \left( \frac{\rho_s Q_s - \rho_w (Q_s - Q_m)}{\rho_m} - \frac{Q_s}{p b} \psi_1 - \frac{Q_m}{b} \psi_2 \right). \]

Based upon Equations 25 and 26, the three variables \( v, \ h, \) and \( z \) can be solved numerically on the electronic digital computer. Furthermore, every solution of this set will be a solution of the original system given by Equations 10, 11, and 18.

B. Finite Difference Approximations

The characteristic equations can be solved by using the first order finite-difference approximation. That is

\[ \int_{x_0}^{x_1} f(x)dx \approx f(x_0)(x_1-x_0). \]  

(27)

Referring to Figure 2, consider conditions \((v, h, z, x, t)\) known at points 1, 2, and 3. The three characteristic curves \( C_1, C_2, \) and \( C_3 \), passing through points 1, 2, and 3, intersect at point P where conditions are unknown.
Using the linear finite-difference form of Equation 27 to points P and 1, 2, 3, the following equations can be obtained from Equations 21 and 26.

Along the $C_1$-curve:

$$(x_p - x_1) - \lambda_1(t_p - t_1) = 0 \quad (28)$$

$$D_1(v_p - v_1) + E_1(h_p - h_1) + F_1(z_p - z_1) = \left[ g(i_0 - i_{f_1}) - Q_1 - R_1 \right] (t_p - t_1) \quad (29)$$

Along the $C_2$-curve:

$$(x_p - x_2) - \lambda_2(t_p - t_2) = 0 \quad (30)$$

$$D_2(v_p - v_2) + E_2(h_p - h_2) + F_2(z_p - z_2) = \left[ g(i_0 - i_{f_2}) - Q_2 - R_2 \right] (t_p - t_2) \quad (31)$$

Along the $C_3$-curve:

$$(x_p - x_3) - \lambda_3(t_p - t_3) = 0 \quad (32)$$

$$D_3(v_p - v_3) + E_3(h_p - h_3) + F_3(z_p - z_3) = \left[ g(i_0 - i_{f_3}) - Q_3 - R_3 \right] (t_p - t_3) \quad (33)$$

In these equations, the subscripts are used to define the location of the known or unknown quality.

There are two typical ways of using Equations 28 to 33 to obtain an approximate numerical solution to the original set of partial differential equations; i.e., (i) Grid of characteristics, (ii) Specified time intervals.

The solution can be simplified if a specified time interval, $\Delta t$, is used in the t-direction, such that
$t_p - t_1 = t_p - t_2 = t_p - t_3 = \Delta t$.

Equal $x$-increments, $\Delta x$, are also used in the $x$-direction. The sizes of $\Delta x$ and $\Delta t$ are selected in such a way that

$$\frac{\Delta x}{\Delta t} \geq \lambda$$

(34)

Referring to Figure 3, $t_i$ and $t_{i+1}$ are the beginning and the end of the time interval $\Delta t$, and $A$, $C$, and $B$ are three adjacent points on the line $t = t_i$; $\Delta x$ apart from each other. Let the intersection of three characteristics, $P$, fall on the intersection of the normal grid lines passing through $t = t_{i+1}$ and $x = x_C$. Denoting the points where the three characteristics intersect the grid line $t = t_i$ as 1, 2, and 3, it is obvious that none of the three points should fall outside of segment $AB$.

**FIGURE 2. INTERSECTION OF THREE CHARACTERISTICS.**

**FIGURE 3. THE SPECIFIED TIME INTERVAL.**
C. **Linear Interpolations**

In the method of specified time intervals, \( x_p \) and \( t_p \) are assigned definite values throughout the computation. This leaves only the three unknowns \( v_p, h_p, \) and \( z_p \) to be determined. From Equations 28, 30, and 32, remembering \( x_p = x_C, t_1 = t_2 = t_3 = t_C; x_1, x_2, \) and \( x_3 \) can be readily evaluated.

\[
x_1 = x_p - \lambda_1(t_p - t_1) \tag{35}
\]
\[
x_2 = x_p - \lambda_2(t_p - t_2) \tag{36}
\]
\[
x_3 = x_p - \lambda_3(t_p - t_3) \tag{37}
\]

Here, because of the sufficiently short distances AC and BC, the \( \lambda \)'s are computed from Equation 24 by using the conditions at point C. The other conditions at points 1, 2, and 3 can then be evaluated in terms of the known conditions at A, B, and C by linear interpolation. That is

\[
\frac{x_C - x_1}{x_C - x_A} = \frac{v_C - v_1}{v_C - v_A} = \frac{h_C - h_1}{h_C - h_A} = \frac{z_C - z_1}{z_C - z_A}. \tag{38}
\]

By use of Equations 28 and 38, recognizing that \( x_C - x_A = x, \)

\[
v_1 = v_C - \lambda_1(v_C - v_A)\Theta, \tag{39}
\]
\[
h_1 = h_C - \lambda_1(h_C - h_A)\Theta, \tag{40}
\]
\[
z_1 = z_C - \lambda_1(z_C - z_A)\Theta. \tag{41}
\]

Similarly, interpolated values are obtained for \( v_2, h_2, z_2, \)
\( v_3, h_3, \) and \( z_3. \)

\[
v_2 = v_C - \lambda_2(v_C - v_A)\Theta, \tag{42}
\]
\[ h_2 = h_C - \lambda_2(h_C - h_A)\Theta, \]  
\[ z_2 = z_C - \lambda_2(z_C - z_A)\Theta, \]  
\[ v_3 = v_C - \lambda_3(v_C - v_B)\Theta, \]  
\[ h_3 = h_C - \lambda_3(h_C - h_B)\Theta, \]  
\[ z_3 = z_C - \lambda_3(z_C - z_B)\Theta. \]

In these equations \( \Theta \) is the grid-mesh ratio, i.e.,
\[ \Theta = \frac{\Delta t}{\Delta x}. \]

**D. Computation of Interior Points**

Equations 29, 31, and 33 can be rearranged as
\[ D_1 v_p + E_1 h_p + F_1 z_p = S_1, \]  
\[ D_2 v_p + E_2 h_p + F_2 z_p = S_2, \]  
\[ D_3 v_p + E_3 h_p + F_3 z_p = S_3, \]
where
\[ S_i = \left\{ g(i_o - i_f) - Q_i - R_i \right\}(t_p - t_i) + D_i v_i + E_i h_i \]
\[ + F_i z_i, \quad i = 1, 2, 3. \]

Solving Equations 48, 49, and 50 simultaneously by determinant or elimination, \( v_p, h_p, \) and \( z_p \) are found as
\[ v_p = \frac{(S_1 F_2 - S_2 F_1)(E_2 F_3 - E_3 F_2) - (S_2 F_3 - S_3 F_2)(E_1 F_2 - E_2 F_1)}{(D_1 F_2 - D_2 F_1)(E_2 F_3 - E_3 F_2) - (D_2 F_3 - D_3 F_2)(E_1 F_2 - E_2 F_1)}, \]
\[ h_p = \frac{(S_1 F_2 - S_2 F_1) - (D_1 F_2 - D_2 F_1)v_p}{(E_1 F_2 - E_2 F_1)}, \]
\[ z_p = \frac{S_1 - D_1 v_p - E_1 h_p}{F_1}. \]
With Equations 35 through 53, the numerical computation can be carried out step by step with the known values of \( v \), \( h \), and \( z \) at points \( A_0, A_1, A_2, \ldots, A_n \) along the \( t = t_0 \) line in Figure 4. The new values of \( v \), \( h \), and \( z \) at points \( B_1, B_2, \ldots, B_{n-1} \) along \( t = t_1 \), can be evaluated similarly by using Equations 35 through 53. With the computed values as knowns, the computation can proceed one more step to achieve the values at \( C_2, \ldots, C_{n-2} \), and so forth.

![Diagram of computational procedure]

**FIGURE 4. COMPUTATIONAL PROCEDURE.**

The number of known points on the \( t = t_0 \) line will determine how far the computation can proceed. However, if suitable boundary conditions are given at \( x = x_0 \) and \( x = x_n \), \( v \), \( h \), and \( z \) at those points marked by a cross in Figure 4 can be calculated, and the computation can be carried out as far as desired with respect to time.
E. **BOUNDARY CONDITIONS**

At the left-end boundary of a long reach, only one characteristic curve (Figure 5) is available in the three unknowns \( v_p, h_p, \) and \( z_p \), two auxiliary equations are needed that specify \( v_p, h_p, z_p \), or some relations between them, so that three equations in three unknowns are available.

At the right-end boundary, two characteristic curves (Figure 6) may be used, and hence, one additional condition should be provided for the computation.

In the following sample computation of sediment deposit at a tide-affected reach, the boundary conditions used are that the flow is steady at the upstream end of the estuary; and that the surface elevation of water at the downstream end changes in the form of a simply sinusoidal function.

F. **THE PROCEDURES IN COMPUTATION**

Steps for the computer solution of transient flow in a channel of nonuniform width by the method of characteristics are stated briefly as follows:

(a) Read-in of initial values, channel geometry and other parameters including \( \Delta x \) and \( \Delta t \).

(b) Calculation of some constants, indices and various variables.

(c) Calculation of conditions at points 1, 2, and 3 by using linear interpolations, or more specifically, Equations 28 through 33.
FIGURE 5. THE LEFT END BOUNDARY.

FIGURE 6. THE RIGHT END BOUNDARY.
(d) Calculation of \( v, h, \) and \( z \) values on \( t = t_{i+1} \) line except those on boundary.

(e) Calculation of the unknown boundary values by the given boundary conditions.

(f) Print-out of results.

(g) Repeat steps b through f as far as desired.

With these brief steps, a FORTRAN IV computer program for the computation of flow in a tidal reach is shown in the Appendix A.

G. **SAMPLE COMPUTATION**

For the computer program, the numerical data used are as follows: the normal flow depth at the upstream end of the estuary \( h = 7.00 \) m; the normal flow velocity \( v = 6.338 \) m/sec; the slope of original fixed bed \( i_o = 0.0003 \); the volume of solids in unit volume of bed sediment \( p = 0.528 \); the fall velocity of sediment \( w = 0.002 \) m/sec; the Manning's \( N = 0.01 \); the coefficient of sediment-transport capacity \( K = 0.20/2650 \); \( \gamma_w = 1000 \) kg/m\(^3\); \( \gamma_s = 2650 \) kg/m\(^3\); the channel width at the upstream boundary = 100 m; and the change of the channel width in the \( x \)-direction \( b_X = 0.001 \). The values used for the parameters \( m \) and \( n \) are \( m = 3 \) and \( n = -1 \). Throughout this computation, \( K \) and \( p \) are assumed as constants. The boundary conditions used are that the flow is steady at the upstream end of the estuary; and that the surface elevation of water at the downstream end changes in the form of a simple sine
function. The amplitude of the tidal wave in the estuary is assumed to be one meter and the period of the tidal wave is eight hours.
Results indicating the change of river configurations in a rectangular channel of variable width were obtained by using the time interval, \( \Delta t \), to be one minute, and the length interval, \( \Delta x \), to be one kilometer. These two values were chosen according to the limitation imposed on the characteristic grid (6). Otherwise, the solution will not converge. Owing to the limitation of space, only a portion of the computed results are listed in Table I. From these data, the flow conditions versus time curves are plotted in Figure 7 to Figure 10. These figures clearly show that the curves are very close to a sinusoidal curve and become stable as the time increases. At the upstream end, the water surface elevation is kept unchanged and the flow velocity, depth, and deposition are computed on the characteristic curve. Therefore, the general trend of flow depth is just opposite to that of sediment deposition. The figures also show that the depth of sediment deposit increases as the flow velocity decreases. This phenomenon can be observed in natural rivers — sediment deposit occurs when the flow velocity decreases. It is noted that at the earlier period of the cycle, the sediment deposit at the downstream end increases with the flow velocity. This seems in contradiction to the previous statement; however, the flow velocity is not the only factor affecting the amount of deposit — the flow depth also
## TABLE I
### COMPUTED RESULTS

<table>
<thead>
<tr>
<th>Station</th>
<th>Time (hr)</th>
<th>Velocity (m/sec)</th>
<th>Depth (m)</th>
<th>Deposition (m)</th>
</tr>
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<td>7.000</td>
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FIGURE 7. VARIATION OF FLOW CONDITIONS WITH TIME, ALONG THE REACH, FOR THREE CHARACTERISTICS.
FIGURE 8. VARIATION OF FLOW CONDITIONS WITH TIME, AT UPSTREAM, FOR THREE CHARACTERISTICS.
FIGURE 9. VARIATION OF FLOW CONDITIONS WITH TIME, AT MID-SECTION, FOR THREE CHARACTERISTICS.
FIGURE 10. VARIATION OF FLOW CONDITIONS WITH TIME, AT DOWNSTREAM, FOR THREE CHARACTERISTICS.
affects the deposit of sediment.

The maximum deposits are 0.35, 0.59, and 0.65 meters at the upstream, the mid-section, and the downstream end of the reach. These values occur, respectively, at \( t = 4 \), 4, and 3 hours from the starting time of the transient flow. The deposit at the downstream end becomes nearly zero, while at the upstream end, it becomes the maximum scour of 0.34 meters at the end of the 8-hr period. Furthermore, it can be clearly seen that the maximum depth of deposit is consistently larger than the maximum depth of scour throughout the entire reach. Speculating from this sample computation, it can be said that the simple-harmonic tidal motion seemingly raises sedimentary problems more from sediment deposit rather than scouring.

For the present study, the values of \( m \) and \( n \) in the equation of sediment concentration were taken as \( m = 3 \) and \( n = -1 \) in order to compare with the results obtained by Chang and Richards. With regard to these two parameters, recent studies of Engelund and Hansen (7) indicate that \( m = 1 \) and \( n = 0.5 \), while Albertson and Garde give \( m = 3 \) and \( n = -2 \), and Velikanov (2,3,4,8) gives \( m = 3 \) and \( n = -1 \). For the best results, these parameters must be obtained from the study of actual data collected in the river under investigation.

Chang and Richards used the same time interval and length interval in their computations. The results which they obtained are shown in Figure 11 through Figure 14. In these
Figure 11. Variation of flow conditions with time, along the reach, for two characteristics.
FIGURE 12. VARIATION OF FLOW CONDITIONS WITH TIME, AT UPSTREAM, FOR TWO CHARACTERISTICS.
FIGURE 13. VARIATION OF FLOW CONDITIONS WITH TIME, AT MID-SECTION, FOR TWO CHARACTERISTICS.
FIGURE 14. VARIATION OF FLOW CONDITIONS WITH TIME, AT DOWNSTREAM, FOR TWO CHARACTERISTICS.
figures, the maximum deposits which occur at the upstream end, the mid-section, and the downstream end of the reach are 0.37, 0.6, and 0.62 meters for \( t = 5, 4, \) and 3 hours. The deposit at the downstream end also becomes nearly zero, while at the upstream end, it becomes the maximum scour of 0.27 meters at the end of the 8-hr period.

These two methods of solution really do not give the same results because of the different assumptions being made, but the shape and trend of the curves indicate good agreement. The time used for computer solution are 33 minutes in this study and 12 minutes for the simplified solution. This is to be expected since more interpolation points were executed and many other details were retained in this study for greater accuracy.
V. SUMMARY AND CONCLUSIONS

The riverbed change in the transient flow can be mathematically simulated by solving three basic partial differential equations dealing with continuity of mass (sediment and water) and momentum. The exact solution is not possible but numerical method may be used for approximate solution. From the foregoing chapters of this study the following summary and conclusions may be drawn:

1. The basic partial differential equations for unsteady flow of homogeneous density, which contain nonlinear terms have been developed for a channel of variable channel width. In the equation of motion, the change of momentum induced by the sediment deposition, the pressure difference induced by the change of sediment concentration along the channel are considered. The general equations are then solved by the method of characteristics, with the aid of a electronic digital computer. Because the effect of the mutually interfering phenomena between the change of the riverbed and the water surface configuration is taken into account for the derivation of the three basic equations, the flow velocity, \( v \), the flow depth, \( h \), and the thickness of sediment deposit, \( z \), can be solved simultaneously.

2. In this study, the simplifications being made are: (1) that the variation of quantities in the vertical direction is neglected; (2) that the change of bed form due to the
effect of bedload is neglected; (3) that the diameters for sediment particles are uniform; and (4) that the sediment concentration in a cross section is assumed to be proportional to the sediment-transport capacity.

3. From the computed data, the velocity, depth, and deposition following the time variation can be traced for any fixed point along the reach, or how velocity, depth, and deposition are distributed along the reach can be observed.

4. Although the plotted curves obtained from the computer output for various points along the channel do not coincide with the simplified solutions, the shape and trend obtained by the two methods of solution indicate favorable agreement.

5. The pattern of sediment deposit is affected by the parameters in sediment-transport equation. By adjusting these parameters, it is possible to simulate the phenomena of transient flow in alluvial channel to closely agree with actual observations in natural rivers.

6. Due to many iterations and more interpolations, the computer time used for the method of three characteristic curves is more than that used for the method of two characteristic curves. However, much computer time can be saved by improving the numerical method for solving the characteristic equation.
(7) In the characteristic method, although closer and closer approximations can be obtained by using the smaller x-increment, this generally increases the computer time.
APPENDIX A

COMPUTER PROGRAM
TSAI YIH-HUA
SOUTH DAKOTA STATE UNIVERSITY
COMPUTATION OF SEDIMENT DEPOSIT IN RIVER ESTUARIES
BY THE METHOD OF CHARACTERISTICS

DIMENSION V(100), H(100), SB(100), TH(100), Z(100),
1 EL(100), BEL(100), SM(100), C(100), PA(100),
2 PL(100), PC(100)

DIMENSION VL(100,10), HL(100,10), ZE(100,10), YD(100,
1 10), FA1(100,10), FA2(100,10), D(100,10),
2 E(100,10), F(100,10), Q(100,10), R(100,10),
3

READ(11,100) N
READ(11,101) GS, CW, GS, GM, SM, SN, P, W
READ(11,102) FN, SO, VX, HO, DQ, G, E, SX, DX
READ(11,103) FL, IMA, DT, PL, 01
READ(11,104) AK, EPSN

STEADY STATE CALCULATIONS

NU=100-N;
NUP=NU/10
DO 10 I=NU,100
SB(I)=V(I) & E
WRITE(12,201) 1, SB(I)
VA=(DG*FN)/SO(I)
H(I)=(VA**(3./5.))/(SU**(3./10.))

V(I)=DG/(H(I)**3.**S(I))
EL(I)=FLOAT(100-I)**SO*DX
BEL(I)=EL(I)

Z(I)=0.

10 TH(I)=H(I)
HEAD=H(100)
WRITE(12,200)
T=0.
IU=0
JA=0

11 TH=1/60.
JA=JA&1
WRITE(12,202) TH
DO 12 I=NU,100
12 WRITE(12,203) 1, V(I), H(I), TH(I), Z(I), BEL(I)

13 T=T+DT
IU=1&IU
IF(T-IMAX) 14, 14, 1000

COEFFICIENTS OF THE THIRD ORDER POLYNOMIAL EQUATION

14 DO 26 I=NU,100
C(I)=AK/(G*H(I)**3)*SM*H(I)**SN
CHARACTERISTIC CURVES

PP=P1
GQ=Q1
IT=0
15 B1=A1-PP
   B2=A2-PP*B1-G2
   B3=A3-PP*B2-Q0+B1
   C1=B1-PP
   C2=B2-PP+C1-Q0
CBARL=C2-B2
DEN=C1*C1-CBARL
IF(DEN) 17,16,17
16 WRITE(12,204)
   GC TC 1000
17 DELTP=(G2*C1-P3)/DEN
   DELTQ=(B3*C1-B2*CBARL)/DEN
   PP=PP&DELTQ
   GQ=Q0&DELTQ
   ABSBP=ABS(DELTQ)
   ABSLO=ABS(DELTQ)
   SUM=ABSBP&ABSLQ
   IT=IT+1
   IF((IT-1) 1000,18,19
18 SUM1=SUM
   GO TO 24
19 IF((IT-5) 24,22,20
20 IF(IT-30) 24,21,21
21 WRITE(12,205) PP,QQ,SM
GO TO 1000
22 IF(SUM-SUN1) 24,23,23
23 WRITE(12,206)
GO TO 1000
24 IF(SUM-EPSSUN) 25,25,15
25 YD(1,1)=(-1.1)*S1
YD(1,2)=-PP/2.6*SQRT(PP*PP/4.-QQ)
YD(1,3)=-PP/2.-SQRT(PP*PP/4.-QQ)
26 CONTINUE
C
C LINEAR INTERPOLATIONS
C
DO 43 I=NU,100
   IF(I-100) 27,28,28
27 JP=5
   GO TO 29
28 JP=2
DO 43 J=1,J0
   XD=YD(I,J)
   NDIL=X0*CT/LX
   JJ=I-NDIL
   DLIL=NDIL
   IF(J-3) 30,31,31
30 UIL=DLIL1.
   GO TO 32
31 UIL=1.-DLIL
32 IF(JJ-MJ) 33,34,34
33 VL(I,J)=V0
   HL(I,J)=H0
   ZL(I,J)=0.0
   GO TO 40
34 IF(J-3) 39,35,35
35 IF(JJ-J9) 36,36,41
36 IF(XD) 37,37,38
37 VL(I,J)=UIL*V(JJ)+DLIL*V(JJ+1)+E(V(JJ)-V(JJ+1))*XD*CT/CX
   HL(I,J)=UIL*H(JJ)+DLIL*H(JJ+1)+E(H(JJ)-H(JJ+1))*XD*CT/CX
   ZL(I,J)=UIL*Z(JJ)+DLIL*Z(JJ+1)+E(Z(JJ)-Z(JJ+1))*XD*DT/CX
   GO TO 40
38 UIL=DLIL1.
39 IF(I-NU) 40,40,41
40 VL(I,J)=UIL*V(JJ)-DLIL*V(JJ-1)-(V(JJ)-V(JJ-1))*XD*DT/CX
   HL(I,J)=UIL*H(JJ)-DLIL*H(JJ-1)-(H(JJ)-H(JJ-1))*XD*DT/CX
   ZL(I,J)=UIL*Z(JJ)-DLIL*Z(JJ-1)-(Z(JJ)-Z(JJ-1))*XD*DT/CX
   GO TO 42
41 VL(I,J)=UIL*V(JJ)-DLIL*V(JJ-1)-(V(JJ)-V(JJ-1))*XD*DT/CX
   HL(I,J)=UIL*H(JJ)-DLIL*H(JJ-1)-(H(JJ)-H(JJ-1))*XD*DT/CX
   ZL(I,J)=UIL*Z(JJ)-DLIL*Z(JJ-1)-(Z(JJ)-Z(JJ-1))*XD*DT/CX

42 \( C(1) = AK / (G + H) \times VL(I, J) \times SH + HL(I, J) \times SN \)

\( GM(I) = (G5 - GM) / \varepsilon(I) \times GM \)

\( FA(I) = AK \times (G5 - SW) / (W \times GM(I)) \)

\( PC(I) = (P \times \varepsilon(I) \times (1 - P) \times GM) / GM(I) \)

\( FA1 = -S \times Z \times FA(I) \times VL(I, J) \times SH + HL(I, J) \times SN \times YD(I, J) \)

\( FA12 = G \times VL(I, J) - YD(I, J) \times (VL(I, J) - VL(I, J)) \times PC(I) \times * \)

\( 2 \quad HL(I, J) \times SH + YD(I, J) \times SN \)

\( FA13 = YD(I, J) \times (YD(I, J) - VL(I, J)) \times SH(I, J) \times (SN \times 1) \times * \)

\( 1 \quad PB(I) = VL(I, J) \times SH + HL(I, J) \times SN \)

\( FB1 = G \times SN / 2 \times \varepsilon(A(I)) \times VL(I, J) \times SN + HL(I, J) \times SN \)

\( FB12 = (SN \times 1) \times PB(I) \times VL(I, J) \times SN + HL(I, J) \times SN \times (G \times VL(I, J)) \)

\( 1 \quad YD(I, J) - YD(I, J) \times PC(I) \times VL(I, J) / 2 / HL(I, J)) \)

\( FB13 = YD(I, J) - VL(I, J) \times (SN \times 1) \times PB(I) \times VL(I, J) \times SN \times (SN \times 1) \times * \)

\( 1 \quad *HL(I, J) \times SN \)

\( FA1(I, J) = (FA11 - FA12) / FA13 \)

\( FB1(I, J) = (FB11 + FB12) / FA13 \)

\( D(I, J) = 1 \times SN \times PB(I) \times VL(I, J) \times SN \times (SN \times 1) \times H(I) \times SN \times 1 \)

\( 1 \quad *FA1(I, J) \)

\( E(I, J) = (SN \times 1) \times PB(I) \times VL(I, J) \times SN \times HL(I, J) \times SN \times FB1(I, J) \)

\( F(I, J) = FA(I, J) \times SH \times (I, J) - PC(I) \times VL(I, J) / HL(I, J) \)

\( Q(I, J) = B \times G \times X \times (PB(I) \times VL(I, J) \times SN \times 1) \times H(I) \times SN \times 1 \)

\( 1 \quad FA(I, J) \times VL(I, J) \times HL(I, J) \times PB1(I, J) \times 5 \times G \times 2 \)

\( HL(I, J) \times 2 \times G \times (I) \)

\( R(I, J) = (G \times (G \times -T \times ERD) / (G \times HL(I, J))) \times Q(I, J) - R(I, J) \times DT \times 1 \)

\( D(I, J) \times VL(I, J) \times E(I, J) \times HL(I, J) \times FB(I, J) \times ZL(I, J) \)

43 CONTINUE

COMPUTATION OF INTERIOR POINTS

\( DO 44 I = NUP + 99 \)

\( D1 = S(I, 1) \times F(I, 2) - S(I, 2) \times F(I, 1) \)

\( D2 = E(I, 1, 2) \times F(I, 3) - E(I, 1, 3) \times F(I, 2) \)

\( D3 = S(I, 1) \times F(I, 3) - S(I, 3) \times F(I, 1) \)

\( D4 = E(I, 1) \times F(I, 2) - F(I, 2) \times F(I, 1) \)

\( D5 = U(I, 1) \times F(I, 2) - D(I, 2) \times F(I, 1) \)

\( D6 = D(I, 2) \times F(I, 3) - D(I, 3) \times F(I, 2) \)

\( V(I) = (D1 \times D2 - D3 \times D4) / (D5 \times D2 - D6 \times D4) \)

\( H(I) = (D1 - D5 \times V(I)) / D4 \)

\( Z(I) = (S(I, 1) - D(I, 1) \times V(I) - E(I, 1) \times H(I)) / F(I, 1) \)
BOUNDARY CONDITIONS

\[ V(NU) = 2.0 \times V(NU1) - V(NU2) \]
\[ H(NU) = (S(NU,2) - D(NU,2) \times v(NU) - F(NU,2) \times HO) / (E(NU,2)) \]
1 - F(NU,2))
\[ Z(NU) = H0 - H(NU) \]
\[ HC = 0.5 \times S14(t \times 1.14 + 159 / 14400.) \times \text{H2} \]
\[ H(100) = (S(100,1) \times (S(100,2) - HC \times F(100,2)) - D(100,2) \times \]
\[ \times E(100,2)) \]
\[ \times F(100,2)) - D(100,2) \times (E(100,1) - F(100,1)) \]
\[ V(100) = (S(100,1) - HC \times F(100,1) - E(100,1) - F(100,1)) \]
1 - H(100) / G(100,1)
\[ Z(100) = HC - H(100) \]
DO 45 I = 100, 1
BEL(1) = EL(1) & Z(1)
45 TH(1) = H(1) / E(1)
IF(1U/60 = JA) 13, 11, 13
100 FORMAT(15)
101 FORMAT(3F10.3)
102 FORMAT(2F10.4)
103 FORMAT(5F10.2)
104 FORMAT(F12.8, F10.4)
200 FORMAT(1H1, 6X, 3-INU., 10X, 4HVEL., 10X, 4HDEP., 9X, 5HTDEP.,
1 \, 9X, 5HSED1., 9X, 5HLEVE.)
201 FORMAT(1H1, 10X, 5HSG ( , I3, 4H ) = , F10.2, 2X, 6MH METERS)
202 FORMAT(1H1, 5X, 7HLNAME = , F7.2, 2X, 6MMINUTES)
203 FORMAT(1H0, 5X, I3, 2F15.3, 3F15.4)
204 FORMAT(1H0, 3HDIVIDED BY ZERO TRY NEW P AND Q)
205 FORMAT(1H5, 1X, 3HPP = , E14.6, 3HPQ = , E14.6, 4HSSUM = , E14.6)
206 FORMAT(1H1, 3HFC1. DIV. FOR THE ASSUMED P AND Q)
1000 STOP
END
APPENDIX B

TABLE II

SYMBOLS USED IN THE FORTRAN IV STATEMENTS AND THEIR EQUIVALENT NOTATIONS OR DESCRIPTIONS

<table>
<thead>
<tr>
<th>Symbols Used in FORTRAN Statements</th>
<th>Equivalent Conventional Notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>AK</td>
<td>K</td>
</tr>
<tr>
<td>BSUBX</td>
<td>b_x</td>
</tr>
<tr>
<td>C</td>
<td>c</td>
</tr>
<tr>
<td>D, DQ, DT, DX</td>
<td>D_i, q, dt, dx</td>
</tr>
<tr>
<td>E, EL</td>
<td>E_i, Z</td>
</tr>
<tr>
<td>F, FAI, FBI, FN</td>
<td>F_i, \Psi_1, \Psi_2, N</td>
</tr>
<tr>
<td>G, GM, GS, GW</td>
<td>g, \gamma_m, \gamma_s, \gamma_w</td>
</tr>
<tr>
<td>H, HL</td>
<td>h, h_i</td>
</tr>
<tr>
<td>P, PA, PB, PC</td>
<td>p, A, B, C</td>
</tr>
<tr>
<td>Q, QM, QS</td>
<td>Q_i, Q_s, Q_m</td>
</tr>
<tr>
<td>R</td>
<td>R_i</td>
</tr>
<tr>
<td>S, SB, SM, SN, SO</td>
<td>S_i, b, m, n, i_0</td>
</tr>
<tr>
<td>TZERO</td>
<td>\tau_0</td>
</tr>
<tr>
<td>V, VL</td>
<td>v, v_i</td>
</tr>
<tr>
<td>W</td>
<td>w</td>
</tr>
<tr>
<td>YD</td>
<td>\lambda_i</td>
</tr>
<tr>
<td>Z, ZL</td>
<td>z, z_i</td>
</tr>
</tbody>
</table>
**APPENDIX B (CONT'D)**

<table>
<thead>
<tr>
<th>Symbols Used in FORTRAN Statements</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1, A2, A3</td>
<td>Coefficients of characteristic equation</td>
</tr>
<tr>
<td>BEL</td>
<td>Riverbed elevation from the datum</td>
</tr>
<tr>
<td>DIL</td>
<td>Real value corresponding to NDIL</td>
</tr>
<tr>
<td>FL</td>
<td>Reach length</td>
</tr>
<tr>
<td>HO</td>
<td>Uniform flow depth</td>
</tr>
<tr>
<td>IU</td>
<td>Integer (for counting the number of times the computations are executed)</td>
</tr>
<tr>
<td>JA</td>
<td>Integer (for the control of time interval in printing)</td>
</tr>
<tr>
<td>N</td>
<td>Number of equal length intervals</td>
</tr>
<tr>
<td>NDIL</td>
<td>Number of stations between computed point and interpolation point</td>
</tr>
<tr>
<td>T</td>
<td>Time in seconds</td>
</tr>
<tr>
<td>TH</td>
<td>Elevation of water surface from the datum</td>
</tr>
<tr>
<td>TM</td>
<td>Time in minutes</td>
</tr>
<tr>
<td>TMAX</td>
<td>Maximum time period for which calculations are desired</td>
</tr>
<tr>
<td>VO</td>
<td>Uniform flow velocity</td>
</tr>
<tr>
<td>XD</td>
<td>Grib-mesh ratio, Δx/Δt</td>
</tr>
</tbody>
</table>
APPENDIX C

Numerical Method for Polynomial Equations

with Real Coefficients
**Fundamental Principle**

A polynomial equation of any degree with real coefficients can be represented in the form

\[ x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad (B-1) \]

This equation has the following properties:

(a) It has \( n \) roots, single or repeated.

(b) It always contains a single real root if \( n \) is a positive odd integer.

(c) The complex roots present themselves in conjugate complex pairs, say, \( D + Ei \) and \( D - Ei \), where \( D \) and \( E \) are real and \( i = \sqrt{-1} \).

(d) Descartes' rule of signs applies to this equation.

Dividing the left-hand side of Equation B-1 by a quadratic factor \((x^2 + px + q)\) leads to the identity

\[ x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = (x^2 + px + q)(x^{n-2} + b_1x^{n-3} + b_2x^{n-4} + \cdots + b_{k-2}x^{n-(k-2)} + b_{k-1}x^{n-(k-1)} + b_kx^{n-k} + \cdots + b_{n-3} + b_{n-2} + bx + s) = x^{n} + (b_1 + p)x^{n-1} + (b_2 + b_1p + q)x^{n-2} + \cdots + (b_k + pb_{k-1} + qb_{k-2})x^{n-k} + \cdots + (pb_{n-2} + qb_{n-3} + r)x + (qb_{n-2} + s) \quad (B-3) \]

Equating the coefficients, yields
\[ a_k = b_k + p b_{k-1} + q b_{k-2}, \quad k = 1, 2, \ldots, n-2; \quad (B-4) \]
\[ a_{n-1} = r + p b_{n-2} + q b_{n-3}, \quad k = n-1; \quad (B-5) \]
\[ a_n = s + q b_{n-2}, \quad k = n. \quad (B-6) \]

There is no \( x^{n-1} \)-term on the right-hand side of Equation B-2, since this would lead to a \( x^{n+1} \)-term when expanded, and hence \( b_{-1} = 0 \) and \( b_0 = 1 \). All other \( b \)'s, however, vary with \( p \) and \( q \). Therefore, \( b_{n-1} \) and \( b_n \), although they do not exist in Equation B-2, can be evaluated and are used to express \( r \) and \( s \) as functions of \( p \) and \( q \), and the \( b \)'s. Extending Equation B-4 for \( k = n-1 \), we have

\[ a_{n-1} = b_{n-1} + p b_{n-2} + q b_{n-3}, \]

and for \( k = n \), we have

\[ a_n = b_n + p b_{n-1} + q b_{n-2}. \]

From Equations B-5 and B-6, we now have

\[ r = a_{n-1} - p b_{n-2} - q b_{n-3} = b_{n-1} \quad (B-7) \]
\[ s = a_n - q b_{n-2} = b_n + p b_{n-1}. \quad (B-8) \]

The fundamental problem is to find \( p \) and \( q \) such that both \( r \) and \( s \) in Equations B-7 and B-8 become zero. Once \( p \) and \( q \) are known, we can then obtain two roots of the original equation B-1 by solving the quadratic equation. The detailed derivation has been shown in "Numerical Methods and Computers" by Kuo.

**The Steps in the Computation of \( P \) and \( Q \)**

1. Select the initial values for \( p \) and \( q \).
(2) Compute all $b_k$'s from $b_1, \ldots, b_n$ from
\[ b_k = a_k - pb_{k-1} - qb_{k-2} \]
where $b_{-1} = 0$, and $b_0 = 1$.

(3) Compute all $c_k$'s from $c_1, \ldots, c_{n-1}$ from
\[ c_k = b_k - pc_{k-1} - qc_{k-2} \]
where $c_{-1} = 0$, and $c_0 = 1$.

(4) Compute $\overline{c}_{n-1} = c_{n-1} - b_{n-1}$.

(5) Using the values of $c_{n-3}$, $c_{n-2}$, $\overline{c}_{n-1}$, $b_n$ and $b_{n-1}$ obtained in steps 2, 3, and 4, compute $\Delta p$ and $\Delta q$:
\[ \Delta p = \frac{b_{n-1}c_{n-1} - b_n c_{n-3}}{(c_{n-2} - \overline{c}_{n-1} c_{n-3})} \]
\[ \Delta q = \frac{b_n c_{n-2} - b_{n-1} \overline{c}_{n-1}}{(c_{n-2} - \overline{c}_{n-1} c_{n-3})} \]

(6) Increment $p$ and $q$ by the amounts $\Delta p$ and $\Delta q$:
\[ p_{i+1} = p_i + \Delta p \]
\[ q_{i+1} = q_i + \Delta q \]
where $i$ is the number of iterations.

(7) Test for convergence:
\[ M = |\Delta p| + |\Delta q| < \varepsilon. \]
If $M > \varepsilon$, return to step 2 and repeat the process. If $M \leq \varepsilon$, then $p$ and $q$ are satisfactory values of the coefficients in the desired quadratic factor $\left(x^2 + px + q\right)$. 


